Advances in Applied Mathematics 62 (2015) 141–159

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Folded bump diagrams for partitions of classical types

Chao-Ping Dong

Institute of Mathematics, Hunan University, Changsha 410082, China

ARTICLE INFO

Article history:
Received 16 April 2014
Received in revised form 17 September 2014
Accepted 19 September 2014
Available online 15 October 2014

MSC:
primary 05A18

Keywords:
Bijections
Bump diagram
Folded bump diagram
Noncrossing partitions
Nonnesting partitions
Quasi-antichain

ABSTRACT

We introduce folded bump diagrams for B_n, C_n and D_n partitions. They allow us to use the type A methods to handle all other classical types simultaneously. As applications, we give uniform interpretations for two families of bijections between noncrossing and nonnesting partitions, where the first family preserves openers and closers, while the second family preserves the statistics a and μ. Here a is the increasing sequence of the minimal elements of the blocks, and μ is the sizes of these blocks. We also extend the results of Chen, Deng, Du, Stanley and Yan (2007) [5] and Kasraoui and Zeng (2006) [10] concerning the symmetry of partitions of type A to other classical types uniformly.

1. Introduction

A *set partition* π of $[n] := \{1, 2, \ldots, n\}$ is a collection of its non-empty subsets B_1, B_2, ..., B_k which are pairwise disjoint, and such that $[n] = B_1 \cup B_2 \cup \cdots \cup B_k$. We call each B_i a *block* of π. The *type* of π is the integer partition λ of n which has a part equal to the cardinality of B_i for $1 \leq i \leq k$. Let us fix the following total order of $[n]$:

E-mail address: chaoping@hnu.edu.cn.

http://dx.doi.org/10.1016/j.aam.2014.09.008

Fig. 1. The bump diagrams of two A_5 partitions.

$$n > \cdots > 2 > 1. \tag{1}$$

After Section 5.1 of Armstrong [1], we present π by its *bump diagram* defined as follows: place n dots on a line, label them from left to right according to (1), and draw a bump between each pair $j > i$ when j and i are consecutive in the same block of π. For example, Fig. 1 displays the bump diagrams for the A_5 partitions $\{\{6,3\},\{5,2\},\{4,1\}\}$ and $\{\{6,1\},\{5,2\},\{4,3\}\}$. We set $\Delta^+(A_{n-1}) = \{e_j - e_i : 1 \le i < j \le n\}$, and read π as a subset Q_π of $\Delta^+(A_{n-1})$ by sending each bump ji to the root $e_j - e_i$. The subset Q_π is actually a *quasi-antichain* in the sense of Definition 2.1. Note that by Lemma 4.1 of [6], the map $\pi \mapsto Q_\pi$ establishes a bijection between partitions of $[n]$ and quasi-antichains of $\Delta^+(A_{n-1})$. For example, the A_5 partition on the left of Fig. 1 corresponds to the quasi-antichain $\{e_6 - e_3, e_5 - e_2, e_4 - e_1\}$ of $\Delta^+(A_5)$. Since partitions of $[n]$ have intrinsic link with the positive root system $\Delta^+(A_{n-1})$, we also call π an A_{n-1} *partition*, and collect all of them as $\mathcal{P}(A_{n-1})$.

After Rubey and Stump [13], the *openers* $\mathrm{op}(\pi)$ are defined to be the non-maximal elements of the blocks in π, whereas the *closers* $\mathrm{cl}(\pi)$ are defined to be the non-minimal elements of the blocks in π. A *singleton* is the element of a block which has only one element. Note that the complement of $\mathrm{op}(\pi) \cup \mathrm{cl}(\pi)$ in $[n]$ consists of the singletons exactly. For example, both of the partitions in Fig. 1 have openers $\{1,2,3\}$, closers $\{4,5,6\}$, and no singleton. Following Section 2 of Fink and Giraldo [7], we define $a(\pi) = (a_1, \cdots, a_k)$ as the increasing sequence of the minimal elements of the blocks, and let $\mu(\pi) = (\mu_1, \cdots, \mu_k)$ be the sizes of these blocks. For example, both of the A_5 partitions in Fig. 1 have $a = (1,2,3)$ and $\mu = (2,2,2)$.

For any $\pi \in \mathcal{P}(A_{n-1})$, we say that a pair of two bumps (ji, kl) of π form *a nesting* if ji nests over kl—that is $j > k > l > i$. Similarly, we say that they form *a crossing* of π if ji crosses with kl. We let $\mathrm{nen}(\pi)$ (resp. $\mathrm{crn}(\pi)$) be the number of nestings (resp. crossings) in π. Moreover, after [5], we use $\mathrm{cr}(\pi)$ (resp. $\mathrm{ne}(\pi)$) to denote the cardinality of a maximal crossing (resp. nesting) in π. Now an A_{n-1} partition is called *nonnesting* (resp. *noncrossing*) if its nesting (resp. crossing) number equals zero. In Fig. 1, the left partition is nonnesting, while the right one is noncrossing. We collect all the nonnesting (resp. noncrossing) A_{n-1} partitions as $NN(A_{n-1})$ (resp. $NC(A_{n-1})$). It is well-known that $NN(A_{n-1})$ and $NC(A_{n-1})$ have the same cardinality. To be more precise, $\#NN(A_{n-1}) = \#NC(A_{n-1}) = \mathrm{Cat}(A_{n-1})$, where

$$\mathrm{Cat}(A_{n-1}) = \frac{1}{n}\binom{2n}{n-1}$$