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On 1324-avoiding permutations

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ABSTRACT

We give an improved algorithm for counting the number of 1324-avoiding permutations, resulting in 5 further terms of the generating function. We analyse the known coefficients and find compelling evidence that unlike other classical length-4 pattern-avoiding permutations, the generating function in this case does not have an algebraic singularity. Rather, the number of 1324-avoiding permutations of length n behaves as

$$B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g.$$

We estimate $\mu = 11.60 \pm 0.01$, $\sigma = 1/2$, $\mu_1 = 0.040 \pm 0.0015$, $g = -1.1 \pm 0.2$ and $B = 7 \pm 1.3$.

1. Introduction

Let π be a permutation on $[n]$ and τ be a permutation on $[k]$. Then τ is said to occur as a *pattern* in π if for some subsequence of π of length k all the elements of the subsequence occur in the same relative order as do the elements of τ. For example, 1324

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occurs as a pattern in 152364 as 1526 and 1536 as both are in the same relative order as 1324. If a permutation τ does not occur in π, then this is said to be a *pattern-avoiding permutation*, or PAP.

Let $P(z) = \sum_{n\geq 0} p_n z^n$ be the ordinary generating function (OGF) for the number of permutations p_n of length n avoiding the pattern 1324. It is well known that, for the classical, length 4 PAPs, the 24 possible patterns fall into one of three classes [26], called Wilf classes. That is to say, there are three distinct OGFs describing all 24 patterns.

For the sequence 1234 and its associated patterns, in 1990 Gessel [14] showed that the number of length $n > 0$ pattern-avoiding permutations is

$$p_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}. \tag{1}$$

Asymptotically,

$$p_n(1234) \sim \frac{81\sqrt{3}}{16\pi} \cdot 9^n \cdot n^{-4},$$

and the generating function $P_{1234}(x) = \sum_n p_n(1234)x^n$ satisfies the linear ODE

$$\begin{aligned}
&\left(9x^5 - 19x^4 + 11x^3 - x^2\right) \cdot \frac{d^3 P_{1234}(x)}{dx^3} + \left(72x^4 - 153x^3 + 90x^2 - 9x\right) \cdot \frac{d^2 P_{1234}(x)}{dx^2} \\
&\quad + \left(126x^3 - 264x^2 + 154x - 16\right) \cdot \frac{dP_{1234}(x)}{dx} \\
&\quad + \left(32 - 72x + 36x^2\right) \cdot P_{1234}(x) = 0,
\end{aligned} \tag{2}$$

with initial conditions $P_{1234}(0) = 1$, $P'_{1234}(2) = 0$, $P''_{1234}(2) = 12$.

For the sequence 1342 and its associated patterns, in 1997 Bóna [3] showed that the number of length $n > 0$ pattern-avoiding permutations is

$$\begin{aligned}
p_n(1342) = (-1)^{n-1} \cdot \frac{(7n^2 - 3n - 2)}{2} \\
+ 3\sum_{k=0}^{n} (-1)^{n-i} \cdot 2^{i+1} \cdot \frac{(2i-4)!}{i!(i-2)!} \cdot \binom{n-i+2}{2}.
\end{aligned} \tag{3}$$

The generating function $P_{1342}(x) = \sum_n p_n(1342)x^n$ satisfies the linear ODE

$$\begin{gathered}
\left(8x^2 + 7x - 1\right) \cdot \frac{d^2 P_{1342}(x)}{dx^2} + (28x - 8) \cdot \frac{dP_{1342}(x)}{dx} + 12 \cdot P_{1342}(x) = 0, \\
P_{1342}(0) = 1, \quad P'_{1342}(0) = 1.
\end{gathered} \tag{4}$$

Indeed, it can be exactly solved to give [22] the simple algebraic expression

$$P_{1342}(x) = \frac{32x}{1 + 20x - 8x^2 - (1-8x)^{3/2}} = \frac{(1-8x)^{3/2}}{2(1+x)^3} + \frac{(1 + 20x - 8x^2)}{2(1+x)^3},$$