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Results on the regularity of square-free monomial ideals

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In a 2008 paper, the first author and Van Tuyl proved that the regularity of the edge ideal of a graph G is at most one greater than the matching number of G. In this note, we provide a generalization of this result to any square-free monomial ideal. We define a 2-collage in a simple hypergraph to be a collection of edges with the property that for any edge E of the hypergraph, there exists an edge F in the 2-collage such that $|E \setminus F| \leq 1$. The Castelnuovo–Mumford regularity of the edge ideal of a simple hypergraph is bounded above by a multiple of the minimum size of a 2-collage. We also give a recursive formula to compute the regularity of a vertex-decomposable hypergraph. Finally, we show that regularity in the graph case is bounded by a certain statistic based on maximal packings of nondegenerate star subgraphs.

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1. Introduction

Let k be a field. There is a natural correspondence between square-free monomial ideals in $R = k[x_1, \dots, x_n]$ and simple hypergraphs over the vertices $V = \{x_1, \dots, x_n\}$. This correspondence has evolved to become an instrumental tool in an active research program in combinatorial commutative algebra – we recommend any of [12,17,19,23] for an overview. One objective of this research program is to relate algebraic properties and invariants of a square-free monomial ideal to combinatorial properties and statistics of the corresponding simple hypergraph. In this note, we will examine the Castelnuovo–Mumford regularity of such ideals, which has been previously studied in works including [5,7,9,14,16,20,22].

In [10, Theorem 6.7], the first author and Van Tuyl showed that the regularity of the edge ideal of a graph G is at most one greater than the matching number of G. Indeed, it follows from their proof and was explicitly noticed in [26] that the regularity of the edge ideal of a graph is at most one greater than the minimum size of a maximal matching. The first objective of this note is to extend this result to the edge ideal of a hypergraph, i.e., to any square-free monomial ideal.

Our first bound for the Castelnuovo–Mumford regularity of a square-free monomial ideal is based on the notion of 2-collage. If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, then a *2-collage for* $\mathcal{H}$ is a subset $\mathcal{C}$ of the edges with the property that for each $E \in \mathcal{E}$ we can delete a vertex v so that $E \setminus \{v\}$ is contained in some edge of $\mathcal{C}$. For uniform hypergraphs, the condition for a collection $\mathcal{C}$ of the edges to be a 2-collage is equivalent to requiring that for any edge E not in $\mathcal{C}$, there exists $F \in \mathcal{C}$ such that the symmetric difference of E and F consists of exactly two vertices. When $\mathcal{H}$ is a graph, it is straightforward to see that for any minimal 2-collage, there is a maximal matching of the same or lesser cardinality. Our first main result is:

Theorem 1.1. *Let $\mathcal{H}$ be a simple d-uniform hypergraph with edge ideal $I \subseteq R$, and let c be the minimum size of a 2-collage in $\mathcal{H}$. Then* $\operatorname{reg}(R/I) \le (d-1)c$.

Indeed, Theorem 1.1 will follow from the following more general result:

Theorem 1.2. *Let $\mathcal{H}$ be a simple hypergraph with edge ideal $I \subseteq R$, and let $\{E_1, \dots, E_c\}$ be a 2-collage in $\mathcal{H}$. Then* $\operatorname{reg}(R/I) \le \sum_{i=1}^{c}(|E_i| - 1)$.

In the case where our hypergraph $\mathcal{H}$ is not a graph, a minimum 2-collage in $\mathcal{H}$ is generally bigger than the *minimax matching number* (that is, the minimum size of a maximal matching). We shall see in Example 3.5 that even in the uniform case, the bound in Theorem 1.1 is no longer true if we replace c by the minimax matching number m of $\mathcal{H}$, and in fact that $\operatorname{reg}(R/I)$ can be arbitrary larger than $(d-1)m$. If $\mathcal{H}$ is a graph, the minimum size of a 2-collage is easily seen to be the minimax matching number, so Theorem 1.1 restricted to graphs recovers [10, Theorem 6.7] and [26, Theorem 11 and discussion following it].

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