Advances in Applied Mathematics 58 (2014) 37–62

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

The structure of 4-flowers of vertically 4-connected matroids ☆

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ARTICLE INFO

Article history:
Received 14 March 2014
Received in revised form 17 April 2014
Accepted 5 May 2014
Available online 28 May 2014

MSC:
52B40

Keywords:
Matroids
Flowers
Connectivity

ABSTRACT

Aikin and Oxley (2012) [2] studied the structure of 4-flowers in 4-connected matroids. In the paper we consider 4-flowers in vertically 4-connected matroids. There is a natural relation of equivalence on such 4-flowers. We characterize the structure that arises when 4-flowers are equivalent under the relation.

1. Introduction

In the paper we study 4-flowers in vertically 4-connected matroids. That extends work of Aikin and Oxley [2]. In particular, we describe the structure that arises when 4-flowers are equivalent.

Before stating our main theorem, we need some technical preliminaries. The matroid terminology used here mainly follows Oxley [5]. Let E be the ground set of a matroid M.

☆ The research was partially supported by CNNSF (No. 11201076), SRFDP (No. 20113514120010), JA11032 and CSC.

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http://dx.doi.org/10.1016/j.aam.2014.05.001
0196-8858/

The *local connectivity* of two subsets X and Y of E, denoted by $\sqcap(X,Y)$, is defined by $\sqcap(X,Y) = r(X) + r(Y) - r(X \cup Y)$. If (X,Y) is a partition of E, then we say $\lambda(X) = \sqcap(X, E-X)$. Let $k \geq 1$ be an integer. If $\lambda(X) \leq k-1$, then we say that X or $(X, E-X)$ is *k-separating*. If $\lambda(X) = k-1$, then X or $(X, E-X)$ is *exactly k-separating*. If $1 \leq k \leq \ell$, then we set $[\ell] = \{1, 2, \cdots, \ell\}$ and $[k, \ell] = \{k, k+1, \cdots, \ell\}$. An ordered partition $\Phi = (P_1, P_2, \cdots, P_n)$ of the ground set E of a matroid M is a *k-flower* with *petals* $P_1, P_2, \cdots, P_n$ if each P_i is exactly k-separating and, when $n \geq 3$, each $P_i \cup P_{i+1}$ is exactly k-separating. When M is 3-connected, a 3-flower is what is defined in [6] as a flower. Assume that I is a proper non-empty subset of $[n]$. Then Φ is a *k-anemone* if $P_I = \bigcup_{i \in I} P_i$ is exactly k-separating for all such I; and Φ is a *k-daisy* if P_I is exactly k-separating for precisely those subsets I whose numbers form a consecutive set in the cyclic order $(1, 2, \cdots, n)$. Aikin and Oxley [1] generalized a result of [6] by showing that every nontrivial k-flower is either a k-anemone or a k-daisy. For an arbitrary petal P_i of Φ, when Φ is a k-anemone, any petal is adjacent to P_i *up to labels*; and when Φ is a k-daisy, a petal is adjacent to P_i *up to labels* if it is adjacent to P_i in the cyclic order $(P_1, P_2, \cdots, P_n)$.

Let X be an exactly 3-separating set of a 3-connected matroid M. The *full closure* of X, denoted by $\mathrm{fcl}(X)$, is the minimum set containing X such that no element $e \in E - \mathrm{fcl}(X)$ satisfies $\lambda(\mathrm{fcl}(X) \cup \{e\}) \leq 2$. An element $e \in P_i$ is *loose* if there is some petal P_j adjacent to P_i up to labels satisfying $e \in \mathrm{fcl}(P_j)$. Petal P_i is *loose* if every element of it is loose, otherwise it is *tight*. And Φ is *tight* if every petal of it is tight.

In the rest of this section, all 4-flowers are in a vertically 4-connected matroid with rank at least seven. Let $\Phi = (P_1, \cdots, P_n)$ be a 4-flower. By [1, Lemma 3.4], there is an integer, denoted by $c(\Phi)$, that represents the local connectivity between any pair of consecutive petals. If $n \geq 5$, then by [1, Lemma 3.4], there is an integer, denoted by $d(\Phi)$, that represents the local connectivity between any pair of nonconsecutive petals. If $n = 4$, then $\sqcap(P_1, P_3)$ may be not equal to $\sqcap(P_2, P_4)$. In this paper, we will completely characterize the structure of 4-flowers with equal local connectivities for any two nonconsecutive petals. By [1, Theorems 1.3 and 1.4], there are six types for such 4-flowers, that is, 4-anemones with $c \in \{0, 1, 2, 3\}$ and 4-daisies of type $(2,1)$ or $(1,0)$, where a 4-flower of type (c,d) means a 4-flower with $c(\Phi) = c$ and $d(\Phi) = d$.

For any tight 3-flower Φ with at least four petals of a 3-connected matroid, Oxley, Semple and Whittle [6] completely characterized its structure, which enables us to find all equivalent 3-flowers of Φ. On the other hand, Chen and Xiang [3] recently proved that via an operation "reducing", every 3-connected representable matroid M with at least nine elements can be decomposed into a set of sequentially 4-connected matroids and three special classes of matroids. Moreover, via an operation similar to "reducing", sequentially 4-connected matroids can be decomposed into vertically 4-connected matroids and a special class of matroid. Thus, it is interesting to know more about the structure of 4-flowers of vertically 4-connected matroids. To characterize the structure of tight 4-flowers with at least four petals of a vertically 4-connected matroid with rank at least seven, first we must generalize the definition of full-closure to allow moving rank-2 sets like [2,4]. Here is an example to illustrate this, which is similar to the one given in [2].