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Combinatorics of balanced carries

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ABSTRACT

We study the combinatorics of addition using balanced digits, deriving an analog of Holte's "amazing matrix" for carries in usual addition. The eigenvalues of this matrix for base b balanced addition of n numbers are found to be $1, 1/b, \cdots, 1/b^n$, and formulas are given for its left and right eigenvectors. It is shown that the left eigenvectors can be identified with hyperoctahedral Foulkes characters, and that the right eigenvectors can be identified with hyperoctahedral Eulerian idempotents. We also examine the carries that occur when a column of balanced digits is added, showing this process to be determinantal. The transfer matrix method and a serendipitous diagonalization are used to study this determinantal process.

1. Introduction

This paper studies the combinatorics of "carries" in basic arithmetic, using balanced digits. To begin we describe the motivation for using balanced digits. When ordinary integers are added, carries occur. Consider a carries table with rows and columns indexed

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by digits $0, 1, \cdots, b-1$ (working base b) and a carry at (i, j) if $i + j \geq b$. Thus when $b = 5$, labeling the rows and columns in the order $0, 1, 2, 3, 4$ the carries matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & b \\ 0 & 0 & b & b & b \\ 0 & b & b & b & b \end{pmatrix}.$$

In general there are $\binom{b}{2}$ carries (so 10 when $b = 5$). If digits are chosen uniformly at random, the chance of a carry is $\binom{b}{2}/b^2 = \frac{1}{2} - \frac{1}{2b}$.

The digits $0, 1, \cdots, b-1$ can be thought of as coset representatives for $b\mathbb{Z} \subseteq \mathbb{Z}$ and the carries are cocycles [14]. For b odd (as assumed throughout this paper), consider instead the balanced representatives $0, \pm 1, \cdots \pm (b-1)/2$. One motivation for using balanced representatives is that they lead to fewer carries. For example, when $b = 5$, writing $\bar{j}$ for $-j$, the digits are $\{0, 1, \bar{1}, 2, \bar{2}\}$. Labeling the rows and columns in the order $\bar{2}, \bar{1}, 0, 1, 2$, the carries matrix is

$$\begin{pmatrix} \bar{b} & \bar{b} & 0 & 0 & 0 \\ \bar{b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & b \end{pmatrix}.$$

(For example, $(-2) + (-2) = -4 = -5 + 1$.) Here there are 6 carries versus 10 for the classical choice. For general b, balanced carries lead to $(b^2 - 1)/4$ carries. This is the smallest number possible [2,12].

Balanced digits are elementary but unfamiliar: for example in base 5, 13 is equal to $1\bar{2}\bar{2}$, and -9 is equal to $\bar{2}1$. Negating numbers negates the digits and the sign of the number is the sign of its left-most digit. Balanced digits were introduced in 1726 by Colson [7]; see Cajori [6] or Chapter 4 of Knuth [16] for history and applications.

Of course, balanced digits may be used for addition with larger numbers. For example:

$$\begin{array}{c} \bar{1}011000 \\[6pt] 1\bar{2}122\bar{1}0 \\ \underline{1\bar{1}\bar{1}2221} \\[6pt] 1210\bar{1}11 \end{array}$$

Here the numbers along the top row are the carries. When two numbers are added the possible carries are $0, 1, \bar{1}$. If n numbers are added, the possible carries are $-\lfloor \frac{n}{2} \rfloor, \cdots, \lfloor \frac{n}{2} \rfloor$.

Suppose now that balanced digits base b are used, n numbers are added, and that the digits are chosen uniformly at random in $\{\overline{\frac{b-1}{2}}, \cdots \frac{b-1}{2}\}$. Consider the carries along