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# Sign imbalances of snakes and valley-signed permutations ☆

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ABSTRACT

One of the combinatorial structures counted by the Springer numbers is the set of snakes, which in type $A_n$ is the set of the alternating permutations and in type $B_n$ (or $D_n$) is the set of certain signed permutations. The set of valley-signed permutations, defined by Josuat-Vergès, Novelli and Thibon, is another structure counted by the Springer numbers of type $B_n$ (or $D_n$). In this paper we determine the sign imbalances of these sets of snakes and valley-signed permutations under various inversion statistics $\mathsf{inv}_w$, $\mathsf{inv}_o$, $\mathsf{inv}_s$, $\mathsf{inv}_B$, and $\mathsf{inv}_D$.


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## 1. Introduction

### 1.1. Springer numbers and snakes

The Springer numbers were introduced by Springer [14] in the study of the Coxeter groups. Let $(W, S)$ be an irreducible Coxeter system and $\ell$ be the length function. We say $D(w) := \{s \in S : \ell(ws) < \ell(w)\}$ is the descent set of $w \in W$. The *Springer number* of $(W, S)$ is defined by

$$K(W) := \max_{J \subseteq S} |\mathcal{D}_J|, \tag{1.1}$$

where $\mathcal{D}_J = \{w \in W : D(w) = J\}$. The sequences of Springer numbers of types $A_n$, $B_n$ and $D_n$ $(n \geq 0)$ are respectively

- type $A_n$: $1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50\,521, \ldots$
- type $B_n$: $1, 1, 3, 11, 57, 361, 2763, 24\,611, 250\,737, 2\,873\,041, 36\,581\,523, \ldots$
- type $D_n$: $1, 1, 1, 5, 23, 151, 1141, 10\,205, 103\,823, 1\,190\,191, 15\,151\,981, \ldots$

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. A permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$ is *alternating* if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. Let $\mathbf{E}_n$ be the set of alternating permutations of $[n]$ and it is well known [17] that $|\mathbf{E}_n|$ is the Euler number.

Let $[\pm n] := \{-n, \ldots, -1\} \cup \{1, \ldots, n\}$. A *signed permutation* $\pi = \pi_{-n} \cdots \pi_{-1}\pi_1 \cdots \pi_n$ (in its *one-line notation*) is a bijection of $[\pm n]$ to itself such that $\pi(-i) = -\pi(i)$ for all $i \in [\pm n]$. For simplicity we denote $-i$ by $\bar{i}$. The *window notation* of $\pi$ is $\pi = [\pi_1\, \pi_2 \cdots \pi_n]$. For example, the signed permutation $\pi = 1\,\bar{4}\,\bar{5}\,3\,\bar{2}\,2\,\bar{3}\,5\,4\,\bar{1}$ has the window notation $[2\,\bar{3}\,5\,4\,\bar{1}]$. Throughout this paper, if there is no danger of confusion, we simply write $2\,\bar{3}\,5\,4\,\bar{1}$ instead of $[2\,\bar{3}\,5\,4\,\bar{1}]$.

Let $\mathfrak{S}_n^{\pm}$ denote the set of signed permutations of $[\pm n]$ and $\mathfrak{S}_n^D \subseteq \mathfrak{S}_n^{\pm}$ denote the set of signed permutations $\pi$ with an even number of negatives among $\pi_1, \ldots, \pi_n$. It is known [2] that $\mathfrak{S}_n$, $\mathfrak{S}_n^{\pm}$ and $\mathfrak{S}_n^D$ are respectively combinatorial descriptions of the Coxeter groups of types $A_n$, $B_n$ and $D_n$. A variety of *signed alternating permutations* are given as follows.

- $\mathbf{S}_n := \{\pi \in \mathfrak{S}_n^{\pm} : \pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots\}$,
- $\mathbf{S}_n^0 := \{\pi \in \mathfrak{S}_n^{\pm} : \pi_1 > 0 \text{ and } \pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots\}$,
- $\mathbf{D}_n := \{\pi \in \mathfrak{S}_n^D : \pi_1 + \pi_2 < 0 \text{ and } \pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots\}$.

Arnol'd [1] proved that $\mathbf{E}_n$, $\mathbf{S}_n^0$ and $\mathbf{D}_n$ are equal to some maximum sets $\mathcal{D}_J$ as introduced in Eq. (1.1) for $W$ the Coxeter groups of types $A_n$, $B_n$ and $D_n$, respectively. Therefore, the cardinalities of $\mathbf{E}_n$, $\mathbf{S}_n^0$ and $\mathbf{D}_n$ are the Springer numbers of types $A_n$, $B_n$ and $D_n$, respectively. Arnol'd coined the term *snakes* to describe the (signed) alternating