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Spectra and eigenvectors of the Segre transformation

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ABSTRACT

Given two sequences $\mathfrak{a} = (a_n)$ and $\mathfrak{b} = (b_n)$ of complex numbers such that their generating series can be written as rational functions where the denominator is a power of $1 - t$, we consider their Segre product $\mathfrak{a} * \mathfrak{b} = (a_n b_n)$. We are interested in the bilinear transformations that compute the coefficient sequence of the numerator polynomial of the generating series of $\mathfrak{a} * \mathfrak{b}$ from those of the generating series of $\mathfrak{a}$ and $\mathfrak{b}$. The motivation to study this problem comes from commutative algebra as the Hilbert series of the Segre product of two standard graded algebras equals the Segre product of the two individual Hilbert series. We provide an explicit description of these transformations and compute their spectra. In particular, we show that the transformation matrices are diagonalizable with integral eigenvalues. We also provide explicit formulae for the eigenvectors of the transformation matrices. Finally, we present a conjecture concerning the real-rootedness of the numerator polynomial of the r-th Segre product of the sequence $\mathfrak{a}$ if r is large enough, under the assumption that the coefficients of the numerator polynomial of the generating series of $\mathfrak{a}$ are non-negative.

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1. Introduction

Given a sequence $\mathfrak{a} = (a_n)_{n \geqslant 0}$ of complex numbers, we consider its formal power series $\mathfrak{a}(t) = \sum_{n \geqslant 0} a_n t^n$. We are interested in the case that $\mathfrak{a}(t)$ can be written as

$$\mathfrak{a}(t) = \sum_{n \geqslant 0} a_n t^n = \frac{h_0(\mathfrak{a}) + h_1(\mathfrak{a})t + \cdots + h_{d_\mathfrak{a}-1}(\mathfrak{a})t^{d_\mathfrak{a}-1}}{(1-t)^{d_\mathfrak{a}}}, \tag{1}$$

which is possible if and only if the sequence $\mathfrak{a}$ is given as a polynomial function in n of degree less than $d_\mathfrak{a}$, see e.g., [13, Theorem 4.1.1]. Adapting the notation in [10], we call $\mathfrak{h}(\mathfrak{a}) := (h_0(\mathfrak{a}), h_1(\mathfrak{a}), \ldots, h_{d_\mathfrak{a}-1}(\mathfrak{a}))$ the *h-vector* and $\mathfrak{h}(\mathfrak{a})(t) = h_0(\mathfrak{a}) + \cdots + h_{d_\mathfrak{a}-1}(\mathfrak{a})t^{d_\mathfrak{a}-1}$ the *h-polynomial* of the rational series $\mathfrak{a}(t)$ respectively of the sequence $\mathfrak{a}$. We are interested in the generating function of the *Segre product* $\mathfrak{a} * \mathfrak{b} := (a_n b_n)_{n \geqslant 0}$ of two sequences $\mathfrak{a} = (a_n)_{n \geqslant 0}$ and $\mathfrak{b} = (b_n)_{n \geqslant 0}$. If $\mathfrak{a}$ and $\mathfrak{b}$ can be expressed as polynomials in n of degree less than $d_\mathfrak{a}$ and $d_\mathfrak{b}$, respectively, this generating series, the so-called *Segre series* of the sequences $\mathfrak{a}$ and $\mathfrak{b}$, is given as

$$(\mathfrak{a} * \mathfrak{b})(t) := \sum_{n \geqslant 0} (a_n b_n) t^n = \frac{h_0(\mathfrak{a} * \mathfrak{b}) + \cdots + h_{d_\mathfrak{a}+d_\mathfrak{b}-2}(\mathfrak{a} * \mathfrak{b})t^{d_\mathfrak{a}+d_\mathfrak{b}-2}}{(1-t)^{d_\mathfrak{a}+d_\mathfrak{b}-1}}.$$

Our aim is to study the transformation of the numerator polynomials of $\mathfrak{a}(t)$ and $\mathfrak{b}(t)$ into the numerator polynomial of $(\mathfrak{a} * \mathfrak{b})(t)$. Though, in principal, this is similar to the investigation of the Veronese transformation for formal power series in [5], there are two crucial differences. First, we do not get a single transformation matrix, describing the transformation of the complete h-vector but a transformation for each entry individually. Second, in our situation the considered transformation will be bilinear rather than linear. However, we will see that the transformation matrices for the different coefficients can be described as particular square block submatrices of a larger rectangular matrix. It will turn out that the study of those submatrices can be performed in a coherent way.

Our original motivation for this problem comes from commutative algebra. By the Hilbert–Serre Theorem (see [6, Chapter 10.4]), the Hilbert series $\operatorname{Hilb}(A,t) = \sum_{i \geqslant 0} \dim_k A_i t^i$ of a standard graded k-algebra $A = \bigoplus_{i \geqslant 0} A_i$ is of the form (1), where the degree of the denominator polynomial equals the Krull dimension of A. Given two standard graded k-algebras $A = \bigoplus_{i \geqslant 0} A_i$ and $B = \bigoplus_{i \geqslant 0} B_i$, it is customary to consider the *Segre product* of those algebras, defined by

$$A * B = \bigoplus_{i \geqslant 0} A_i \otimes_k B_i,$$

see e.g., [7,9,11]. Note that if A and B are polynomial rings in r variables, i.e., $A = k[x_1, \ldots, x_r]$ and $B = k[y_1, \ldots, y_r]$, the Segre product $A * B$ can be viewed as the homogeneous coordinate ring of the image of the *Segre embedding*

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