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Using functional equations to enumerate 1324-avoiding permutations

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ABSTRACT

We consider the problem of enumerating permutations with exactly r occurrences of the pattern 1324 and derive functional equations for this general case as well as for the pattern avoidance ($r = 0$) case. The functional equations lead to a new algorithm for enumerating length n permutations that avoid 1324. This approach is used to enumerate the 1324-avoiders up to $n = 31$. We also extend those functional equations to account for the number of inversions and derive analogous algorithms.

1. Introduction

Let $a_1 \dots a_k$ be a sequence of k distinct positive integers. We define the *reduction* of this sequence, denoted by $\mathrm{red}(a_1 \dots a_k)$, to be the length k permutation $\tau = \tau_1 \dots \tau_k$ that is order-isomorphic to $a_1 \dots a_k$ (i.e., $a_i < a_j$ if and only if $\tau_i < \tau_j$ for every i and j). Given a (permutation) pattern $\tau \in \mathcal{S}_k$, we say that a permutation $\pi = \pi_1 \dots \pi_n$ *contains* the pattern τ if there exists $1 \leqslant i_i < i_2 < \dots < i_k \leqslant n$ such that $\mathrm{red}(\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}) = \tau$, in which case we call $\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}$ an *occurrence* of τ. We also define $N_\tau(\pi)$ to be the number

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of occurrences of pattern τ in the permutation π. For example, if the pattern $\tau = 123$, the permutation 53412 avoids the pattern τ (so $N_{123}(53412) = 0$), whereas the permutation 52134 contains two occurrences of τ (so $N_{123}(52134) = 2$).

For a pattern τ and non-negative integer $r \geqslant 0$, we define the set

$$\mathcal{S}_n(\tau, r) := \{\pi \in \mathcal{S}_n\colon\ \pi \text{ has exactly } r \text{ occurrences of the pattern } \tau\}$$

and also define $s_n(\tau, r) := |\mathcal{S}_n(\tau, r)|$. For the $r = 0$ case, we say that the two patterns σ and τ are *Wilf-equivalent* if $s_n(\sigma, 0) = s_n(\tau, 0)$ for all n. Additionally, two patterns that are Wilf-equivalent are said to belong to the same *Wilf-equivalence class*. Note that the $r = 0$ case corresponds with "classical" pattern avoidance, which has been well-studied. Work on the more general problem ($r \geqslant 0$) has usually been restricted to patterns of length 3 and small r.

A little more is known for the pattern avoidance problem, but the problem quickly gets difficult. For each $\tau \in \mathcal{S}_3$, it is well known that $s_n(\tau, 0) = \frac{1}{n+1}\binom{2n}{n}$ (the Catalan numbers) [8]. For the length 4 patterns, there are three cases (Wilf-equivalence classes) to consider: 1234, 1342, and 1324. The enumeration for the 1234-avoiding permutations was solved by Gessel in [7]. Later, Bóna solved the case for the 1342-avoiding permutations in [3]. The pattern 1324, however, has been notoriously difficult to enumerate.

There is currently no non-recursive formula known for computing $s_n(1324, 0)$, and precise asymptotics are not known either. Marinov and Radoičić developed an approach in [10] using generating trees and computed $s_n(1324, 0)$ for $n \leqslant 20$. Another approach (using insertion encoding) was used by Albert et al. in [1] to compute $s_n(1324, 0)$ for $n \leqslant 25$ (five more terms). Given the difficulty of this pattern, Zeilberger has even conjectured that "Not even God knows $s_{1000}(1324, 0)$" [6].

Given the difficulty of exact enumeration, work has also been done on studying how the sequence $s_n(\tau, 0)$ grows for various patterns. We define the *Stanley–Wilf limit* of a pattern τ to be:

$$L(\tau) := \lim_{n\to\infty} \bigl(s_n(\tau, 0)\bigr)^{1/n}. \tag{1}$$

Thanks to results by Arratia [2] and Marcus and Tardos [9], we know that for each pattern τ, the limit $L(\tau)$ exists and is finite. For patterns of length three, the Stanley–Wilf limit is known to be 4. Additionally, Regev [13] showed that $L(1234) = 9$, while Bóna's result in [3] gives us $L(1342) = 8$. The exact limit for the pattern 1324, however, is still unknown. The best known lower bound is by Albert et al. [1], who showed that $L(1324) \geqslant 9.47$. The best known upper bound has seen some improvements in recent years. The recent "best" upper bound was improved by Claesson, Jelínek, and Steingrímsson in [5] to $L(1324) \leqslant 16$. That approach was then refined by Bóna in [4] to show that $L(1324) < (7 + 4\sqrt{3}) \approx 13.93$.

Additionally, Claesson, Jelínek, and Steingrímsson conjectured that the number of length n permutations avoiding 1324 with exactly k inversions was non-decreasing in n