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Counting symmetry classes of dissections of a convex regular polygon

Douglas Bowman, Alon Regev *

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, United States

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ABSTRACT

This paper proves explicit formulas for the number of dissections of a convex regular polygon modulo the action of the cyclic and dihedral groups. The formulas are obtained by making use of the Cauchy–Frobenius Lemma as well as bijections between rotationally symmetric dissections and simpler classes of dissections. A number of special cases of these formulas are studied. Consequently, some known enumerations are recovered and several new ones are provided.

1. Introduction

In 1963 Moon and Moser [13] enumerated the equivalence classes of triangulations of a regular convex n-gon modulo the action of the dihedral group D_{2n}. A year later, Brown [3] enumerated the equivalence classes of these triangulations modulo the action of the cyclic group Z_n. Recall that the triangulations of an n-gon are in bijection with

* Corresponding author.
E-mail addresses: bowman@math.niu.edu (D. Bowman), regev@math.niu.edu (A. Regev).

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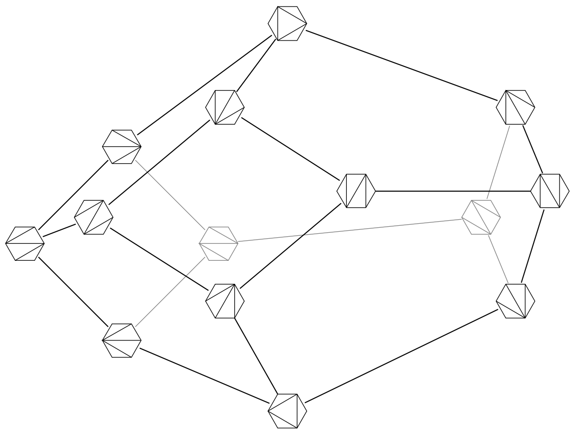

Fig. 1. The three-dimensional associahedron.

the vertices of the associahedron of dimension $n-3$ (see Fig. 1). Lee [10] showed that the associahedron can be realized as a polytope in $(n-3)$-dimensional space having the dihedral symmetry group D_{2n}. Thus Moon and Moser's result and Brown's result are equivalent to enumerating of the vertices of the associahedron modulo the dihedral action and the cyclic action, respectively. The enumeration by Moon and Moser also arose recently in the work of Ceballos, Santos and Ziegler [7]. Their work describes a family of realizations of the associahedron (due to Santos), and proves that the number of normally non-isomorphic realizations is the number of triangulations of a regular polygon modulo the dihedral action. In this paper we generalize the results of Moon and Moser, as well as Brown, and enumerate the number of *dissections* of regular polygons modulo the dihedral and cyclic actions.

Definition 1. Let $n \geqslant 3$. A *k-dissection* of an n-gon is a partition of the n-gon into $k+1$ polygons by k non-crossing diagonals. A *triangulation* is an $(n-3)$-dissection of an n-gon and an *almost-triangulation* is an $(n-4)$-dissection. Let $G(n,k)$ be the set of k-dissections of an n-gon, and let $G(n) = \bigcup_{k=0}^{n-3} G(n,k)$.

In terms of associahedra, a k-dissection corresponds to an $(n-k-3)$-dimensional face on an associahedron of dimension n. A natural generalization of the results of Moon and Moser and of Brown is the enumeration of $G(n,k)/D_{2n}$ and $G(n,k)/Z_n$, the sets of cyclic and dihedral classes, respectively, in $G(n,k)$. In 1978, Read [15] considered an equivalent problem. He enumerated certain classes of cellular structures, which are in bijection with $G(n,k)/D_{2n}$ and $G(n,k)/Z_n$. Read found generating functions for the number of such classes, and included tables of values [15, Tables 3 and 5]. In fact, the first diagonal of Table 5 of Read corresponds to the sequence found by Moon and Moser, and the first diagonal of Table 3 of Read corresponds to the sequence found by Brown. Lisonek [11,12] studied these results of Read and showed that the sequences $|G(n,k)/D_{2n}|$ and $|G(n,k)/Z_n|$ are "quasi-polynomial" in n when k is fixed. (Here and