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Combinatorics of non-ambiguous trees ☆

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ABSTRACT

This article investigates combinatorial properties of non-ambiguous trees. These objects we define may be seen either as binary trees drawn on a grid with some constraints, or as a subset of the tree-like tableaux previously defined by Aval, Boussicault and Nadeau. The enumeration of non-ambiguous trees satisfying some additional constraints allows us to give elegant combinatorial proofs of identities due to Carlitz, and to Ehrenborg and Steingrímsson. We also provide a hook formula to count the number of non-ambiguous trees with a given underlying tree. Finally, we use non-ambiguous trees to describe a very natural bijection between parallelogram polyominoes and binary trees.

1. Introduction

It is well known that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ enumerate many combinatorial objects, such as binary trees and parallelogram polyominoes. Several bijective proofs in the literature show that parallelogram polyominoes are enumerated by Catalan numbers, the two most classical being Delest–Viennot's bijection with Dyck paths [9] and Viennot's bijection with bicolored Motzkin paths [9].

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In this paper we demonstrate a bijection – which we believe is more natural – between binary trees and parallelogram polyominoes. In some sense, we show that parallelogram polyominoes may be seen as two-dimensional drawings of binary trees. This point of view gives rise to a new family of objects – we call them *non-ambiguous trees* – which are particular compact embeddings of binary trees in a grid.

The tree structure of these objects leads to a hook formula for the number of non-ambiguous trees with a given underlying tree. Unlike the classical hook formula for trees due to Knuth (see [11, §5.1.4, Exercise 20]), this one is defined on the edges of the tree.

Non-ambiguous trees are in bijection with permutations such that all their (strict) excedances stand at the beginning of the permutation word. Ehrenborg and Steingrímsson in [10] give a closed formula (involving Stirling numbers of the second kind) for the number of such permutations. We show that this formula can be easily proved using non-ambiguous trees and a variation of the insertion algorithm for tree-like tableaux introduced in [2]. Indeed, non-ambiguous trees can also be seen as a subclass of tree-like tableaux, objects defined in [2] and that are in bijection with permutation tableaux [18] or alternative tableaux [13,19].

A particular subclass of non-ambiguous trees leads to unexpected combinatorial interpretations. We study complete non-ambiguous trees, defined as non-ambiguous trees such that their underlying binary tree is complete, and show that their enumerating sequence is related to the formal power series of the logarithm of the Bessel function of order 0. This gives rise to new combinatorial interpretations of some identities due to Carlitz [6], and to the proof of a related identity involving Catalan numbers, which had been conjectured by P. Hanna.

The paper is organized as follows: in Section 2 we define non-ambiguous trees. Then, in Section 3 we give the enumeration of non-ambiguous trees satisfying certain constraints: those contained in a given rectangular box, and those with a fixed underlying tree. Section 4 introduces the family of complete non-ambiguous trees, studies the relations between this family and the Bessel function, and proves combinatorial identities involving Catalan numbers and the sequence enumerating complete non-ambiguous trees. In Section 5 we describe our new bijection between binary trees and parallelogram polyominoes. To conclude, we present in Section 6 some perspectives related to our study.

2. Definitions and notations

In this paper, trees are embedded in a bidimensional grid $\mathbb{N} \times \mathbb{N}$. The grid is not oriented as usual: the x-axis has south–west orientation, and the y-axis has south–east orientation, as shown in Fig. 1.

Every x-oriented (resp. y-oriented) line will be called column (resp. row). Each column (resp. row) on this grid is numbered with an integer corresponding to its y (resp. x) coordinate. A vertex v located on the intersection of two lines has the coordinate representation: $(X(v), Y(v))$.