Advances in Applied Mathematics 56 (2014) 109–134

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Path sets in one-sided symbolic dynamics

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ARTICLE INFO

Article history:
Received 21 January 2013
Accepted 23 December 2013
Available online 26 March 2014

MSC:
primary 37B10
secondary 28A80, 54H20

Keywords:
Finite automata
Symbolic dynamics
Sofic shift
Follower set
Topological entropy

ABSTRACT

Path sets are spaces of one-sided infinite symbol sequences associated to pointed graphs $(\mathcal{G}, v_0)$, which are edge-labeled directed graphs with a distinguished vertex v_0. Such sets arise naturally as address labels in geometric fractal constructions and in other contexts. The resulting set of symbol sequences need not be closed under the one-sided shift. This paper establishes basic properties of the structure and symbolic dynamics of path sets, and shows that they are a strict generalization of one-sided sofic shifts.

1. Introduction

This paper investigates the concept of a path set, which is a notion in one-sided symbolic dynamics defined below. Path sets form an enlargement of the class of one-sided sofic shifts that includes certain additional closed sets not invariant under the one-sided shift. Let $\mathcal{A}^{\mathbb{N}}$ denote the full one-sided shift space on the finite alphabet $\mathcal{A}$, topologized

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[1] Work of J.L. was supported by NSF grant DMS-1101373.

http://dx.doi.org/10.1016/j.aam.2013.12.005

with the product topology. It is a compact, completely disconnected topological space. Path sets are a distinguished class of closed subsets of $\mathcal{A}^{\mathbb{N}}$ constructed as follows. We are given a finite directed graph G with edges bearing labels from $\mathcal{A}$ with a marked vertex v. A *path set* prescribed by the data (G, v) is the collection of one-sided infinite sequences of edge labels assigned to all infinite paths in the finite directed graph G emanating from the vertex v of G. It is easy to see that path sets are closed subsets of $\mathcal{A}^{\mathbb{N}}$. We denote the collection of all path sets on the alphabet $\mathcal{A}$ by $\mathcal{C}(\mathcal{A})$.

One reason for studying path sets is that they naturally arise in connection with geometric constructions of fractals and limit sets of discrete groups. Associated to these constructions are *address maps*, given by paths in finite directed graphs, which specify labels of points in the limit sets of such recursive constructions. Such address maps arise in geometric graph-directed constructions (Mauldin and Williams [30], Edgar [14, Sec. 4.3]), in iterated function systems (Barnsley [10, Sec. 4.1]), in describing limit sets of various discrete group actions (Mauldin and Urbański [29]) and in describing boundaries of fractal tiles (Akiyama and Lorident [4]). Under some conditions addresses are unique, but in other circumstances multiple addresses label the same geometric point. There are known conditions, such as the "open set condition", under which almost all points have a unique address (Bandt et al. [8]). The symbolic dynamics objects we study here are the complete sets of distinct addresses given by an address map, whether or not addresses are unique.

Path sets are not a new concept; they have previously appeared in the symbolic dynamics literature under the name "follower set", typically as an auxiliary construction. The usual framework of coding in symbolic dynamics, as given in Lind and Marcus [27], restricts to shift-invariant sets (ones with $\sigma(X) \subset X$) and emphasizes two-sided dynamics; shift-invariant sets in the one-sided case are considered in Kitchens [21] for shifts of finite type. Path sets are objects in one-sided dynamics that are not always invariant under the one-sided shift map; the initial condition imposed by the marked vertex typically breaks shift-invariance. Their distinctive properties related to lack of shift invariance seem not to have been studied in any detail. This paper proposes the terminology *path sets*, which is consistent with language used in fractal constructions [14, Sec. 4.3], because "follower set" is used in the symbolic dynamics literature with several different meanings, see the discussion in Section 1.2. An alternate descriptive term for path set could be *pointed follower set*.

The object of this paper is to establish basic properties of path sets in a relatively self-contained manner. It particularly addresses those properties connected to symbolic dynamics and the action on them of the one-sided shift map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ which sends $\sigma(a_0a_1a_2, \ldots) = (a_1a_2a_3, \ldots)$, with each $a_i \in \mathcal{A}$. We show that path sets form a strict generalization of one-sided sofic shifts on the alphabet $\mathcal{A}$, and characterize sofic shifts as exactly the (one-sided) shift-invariant members of $\mathcal{C}(\mathcal{A})$. Members of $\mathcal{C}(\mathcal{A})$ retain the good set-theoretic and topological entropy properties of one-sided sofic shifts. The enlargement from the class of sofic shifts to the class of path sets gives a class closed under a larger set of operations than for sofic shifts; in particular when the alphabet has