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Lattice-supported splines on polytopal complexes

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ABSTRACT

We study the module $C^r(\mathcal{P})$ of piecewise polynomial functions of smoothness r on a pure n-dimensional polytopal complex $\mathcal{P} \subset \mathbb{R}^n$, via an analysis of certain subcomplexes $\mathcal{P}_W$ obtained from the intersection lattice of the interior codimension one faces of $\mathcal{P}$. We obtain two main results: first, we show that in sufficiently high degree, the vector space $C^r_k(\mathcal{P})$ of splines of degree $\leqslant k$ has a basis consisting of splines supported on the $\mathcal{P}_W$ for $k \gg 0$. We call such splines *lattice-supported*. This shows that an analog of the notion of a star-supported basis for $C^r_k(\Delta)$ studied by Alfeld–Schumaker in the simplicial case holds [3]. Second, we provide a pair of conjectures, one involving lattice-supported splines, bounding how large k must be so that $\dim_{\mathbb{R}} C^r_k(\mathcal{P})$ agrees with the McDonald–Schenck formula [14]. A family of examples shows that the latter conjecture is tight. The proposed bounds generalize known and conjectured bounds in the simplicial case.

1. Introduction

Let $\mathcal{P}$ be a subdivision of a region in $\mathbb{R}^n$ by convex polytopes. $C^r(\mathcal{P})$ denotes the set of piecewise polynomial functions (splines) on $\mathcal{P}$ that are continuously differentiable of order r. Study of the spaces $C^r(\mathcal{P})$ is a fundamental topic in approximation theory and numerical analysis (see [8]) while within the past decade geometric connections have been

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made between $C^0(\mathcal{P})$ and equivariant cohomology rings of toric varieties [16]. Practical applications of splines include computer aided design, surface modeling, and computer graphics [8].

A central problem in spline theory is to determine the dimension of (and a basis for) the vector space $C^r_k(\mathcal{P})$ of splines whose restriction to each facet of $\mathcal{P}$ has degree at most k. In the bivariate, simplicial case, these questions are studied by Alfeld and Schumaker in [1] and [2] using Bernstein–Bezier methods. A signature result appears in [2], which gives a dimension formula for $C^r_k(\mathcal{P})$ when $k \geqslant 3r+1$ and $\mathcal{P}$ is a generic simplicial complex. An algebraic approach to the dimension question was pioneered by Billera in [5] using homological and commutative algebra. This method has been refined and extended by Schenck, Stillman, and McDonald ([20] and [14]). The last of these gives a polyhedral version of the Alfeld–Schumaker formula in the planar case, building on work of Rose [17,18] on dual graphs.

In applications it is often important to find a basis of $C^r_k(\mathcal{P})$ which is "locally supported" in some sense. In the simplicial case the natural thing to require is that the basis elements are supported on stars of vertices. Alfeld and Schumaker [3] call such a basis *minimally supported* or *star-supported*, and they show that for $k = 3r+1$ it is not always possible to construct a star-supported basis of $C^r_k(\mathcal{P})$. However in the planar simplicial case the bases constructed for $C^r_k(\Delta)$ in [12] and [13] for $k \geqslant 3r+2$ are in fact star-supported. Alfeld, Schumaker, and Sirvent show in [4] that in the trivariate case $C^r_k(\Delta)$ has a star-supported basis for $k > 8r$.

In this paper, we first show that there are polyhedral analogs of "star-supported splines". Our main technical tool is Proposition 3.5, which utilizes polytopal subcomplexes $\mathcal{P}_W \subset \mathcal{P}$ associated to certain linear subspaces $W \subset \mathbb{R}^n$ to give a precise description of localization of the module $C^r(\mathcal{P})$. We use this description to tie together local characterizations of projective dimension and freeness due to Yuzvinsky [22] and Billera and Rose [7]. A consequence of Proposition 3.5 is that there are generators for $C^r(\mathcal{P})$ as an $R = \mathbb{R}[x_1, \dots, x_n]$-module which are supported on the complexes $\mathcal{P}_W$. From this follows Theorem 4.4, that for $k \gg 0$, $C^r_k(\mathcal{P})$ has an $\mathbb{R}$-basis which is supported on subcomplexes of the form $\mathcal{P}_W$. We call such a basis a *lattice*-supported basis, where the lattice (in the sense of a graded poset) of interest is the intersection lattice of the interior codimension one faces of $\mathcal{P}$. A lattice-supported basis reduces to a star-supported basis in the simplicial case.

In Section 5 we use the *regularity* of a graded module to analyze $\dim_{\mathbb{R}} C^r_k(\mathcal{P})$, which is also done in [21]. We propose in Conjecture 5.4 a regularity bound on the module of locally supported splines in the case $\mathcal{P} \subset \mathbb{R}^2$. If true this conjecture gives a bound for when the McDonald–Schenck formula [14] for $\dim_{\mathbb{R}} C^r_k(\mathcal{P})$ holds which generalizes the Alfeld–Schumaker $3r+1$ bound in the planar simplicial case. We also propose (Conjecture 5.5) a stronger regularity bound on $C^r(\hat{\mathcal{P}})$ for $\mathcal{P} \subset \mathbb{R}^2$ and give a family of examples to show that this proposed bound is tight. Conjecture 5.5 generalizes a conjecture of Schenck that the Alfeld–Schumaker dimension formula [2] holds when $k \geqslant 2r+1$ in the planar simplicial case [19].