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Minkowski tensor density formulas for Boolean models ☆

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A B S T R A C T

A stationary Boolean model is the union set of random compact particles which are attached to the points of a stationary Poisson point process. For a stationary Boolean model with convex grains we consider a recently developed collection of shape descriptors, the so called Minkowski tensors. By combining spatial and probabilistic averaging we define Minkowski tensor densities of a Boolean model. These densities are global characteristics of the union set which can be estimated from observations. In contrast local characteristics like the mean Minkowski tensor of a single random particle cannot be observed directly, since the particles overlap. We relate the global to the local properties by density formulas for the Minkowski tensors. These density formulas generalize the well known formulas for intrinsic volume densities and are obtained by applying results from translative integral geometry. Our results support the idea that the degree of anisotropy of a Boolean model may be expressed in terms of the Minkowski tensor densities. Furthermore we observe that for smooth grains the mean curvature radius function of a particle can be reconstructed from the Minkowski tensor densities. In a simulation study we numerically determine Minkowski tensor densities for non-isotropic Boolean models based on ellipses and on rectangles

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in two dimensions and find excellent agreement with the derived analytic density formulas. The tensor densities can be used to characterize the orientational distribution of the grains and to estimate model parameters for non-isotropic distributions. In particular, the numerically determined values for the density of the Euler characteristic allow the estimation of certain mixed functionals of the grains.

1. Introduction

The Boolean model appeared early in applied probability, typically in attempts to describe random geometrical structures of physics and materials science by overlapping spherical grains. It was Matheron who created the general theory of the Boolean model and studied its basic properties [33]. Let $\{\xi_i\colon i \in \mathbb{N}\}$ be a stationary Poisson point process in $\mathbb{R}^n$ with intensity $\gamma > 0$, and let $K_1, K_2, \dots$ be independent, identically distributed random compact sets with distribution $\mathbb{Q}$, which are independent of the point process $\{\xi_i\colon i \in \mathbb{N}\}$. Then, under the integrability assumption (3) on $\mathbb{Q}$, the union of the translated grains

$$Z := \bigcup_{i=1}^{\infty} (K_i + \xi_i)$$

is a random closed set, which is called the stationary Boolean model with intensity γ and grain distribution $\mathbb{Q}$; see [55] for a review of recent developments in this context.

The Boolean model is a popular model in materials science and the physics of heterogeneous media [57] relating shape to physical properties. Many porous materials are built up by the successive addition of inclusions (grains, pores or cracks) within a background phase [9,3]; such materials can be modeled by a Boolean process. It has been applied, in particular, on foamed materials, ceramic powders [41], wood composites [58], sedimentary rock [54,32], fractured materials or hydrating cement-based materials [10]. Depending on the specific application, either the pore space or the solid phase of a material may be described as a Boolean model Z. For example, the pore space of bread [7] was modeled by the Boolean model, whereas for sintered ceramic composites it is the solid phase which is described as a Boolean model; see [41]. In particular, for the reconstruction of two-phase materials the Boolean model is successful, which finally allows excellent predictions of the shape dependence of thermodynamic quantities [27] and transport properties [3] in porous media.

Various other physical phenomena can be described and studied by the Boolean model including percolation [35,37] and elasticity [4]. Many attempts have been made to predict mechanical properties from structural features; some of these are based on Boolean models and other random set models [57,24]. In practice, measurements taken on samples of