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The shape of random pattern-avoiding permutations

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ABSTRACT

We initiate the study of limit shapes for random permutations avoiding a given pattern. Specifically, for patterns of length 3, we obtain delicate results on the asymptotics of distributions of positions of numbers in the permutations. We view the permutations as 0–1 matrices to describe the resulting asymptotics geometrically. We then apply our results to obtain a number of results on distributions of permutation statistics.

0. Introduction

The *Catalan numbers* is one of the most celebrated integer sequences, so much that it is hard to overstate their importance and applicability. In the words of Thomas Koshy, “Catalan numbers are [...] fascinating. Like the North Star in the evening sky, they are a beautiful and bright light in the mathematical heavens.” Richard Stanley called them “the most special” and his “favorite number sequence” [30]. To quote Martin Gardner, “they

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have the delightful propensity for popping up unexpectedly, particularly in combinatorial problems" [26]. In fact, Henry Gould's bibliography [27] lists over 450 papers on the subject, with many more in recent years.

Just as there are many combinatorial interpretations of Catalan numbers [51, Exc. 6.19] (see also [43,50,52]), there are numerous results on statistics of various such interpretations (see e.g. [8,51]), as well as their probabilistic and asymptotic behavior (see [15,24]). The latter results usually come in two flavors. First, one can study the probability distribution of statistics, such as the expectation, the standard deviation and higher moments. The approach we favor is to define the shape of a large random object, which can be then analyzed by analytic means (see e.g. [3,54,55]). Such objects then contain information about a number of statistics, under one roof.

In this paper we study the set $\mathcal{S}_n(\pi)$ of permutations $\sigma \in S_n$ *avoiding a pattern* π. This study was initiated by Percy MacMahon and Don Knuth, who showed that the size of $\mathcal{S}_n(\pi)$ is the Catalan number C_n, for all permutations $\pi \in S_3$ [33,35]. These results opened a way to a large area of study, with numerous connections to other fields and applications [31] (see also Section 8.2).

We concentrate on two classical patterns, the **123**- and **132**-avoiding permutations. Natural symmetries imply that other patterns in S_3 are equinumerous with these two patterns. We view permutations as 0–1 matrices, which we average, scale to fit a unit square, and study the asymptotic behavior of the resulting family of distributions. Perhaps surprisingly, behavior of these two patterns is similar on a small scale (linear in n), with random permutations approximating the reverse identity permutation $(n, n-1, \ldots, 1)$. However, on a larger scale (roughly, on the order n^α away from the diagonal), the asymptotics of shapes of random permutations in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, are substantially different. This explains, perhaps, why there are at least nine different bijections between two sets, all with different properties, and none truly "ultimate" or "from the book" (see Section 8.4).

Our results are rather technical and contain detailed information about the random pattern avoiding permutations, on both the small and large scale. We exhibit several regimes (or "phases"), where the asymptotics are unchanged, and painstakingly compute the precise limits, both inside these regimes and at the phase transitions. Qualitatively, for **123**-avoiding permutations, our results are somewhat unsurprising, and can be explained by the limit shape results on the *Brownian excursion* (see Section 8.8); still, our results go far beyond what was known. However, for the **132**-avoiding permutations, our results are extremely unusual, and have yet to be explained even on a qualitative level (see Section 8.9).

The rest of the paper is structured as follows. In the next section we first present examples and calculations which then illustrate the "big picture" of our results. In Section 2 we give formal definitions of our matrix distributions and state basic observations on their behavior. We state the main results in Section 3, in a series of six theorems of increasing complexity, for the shape of random permutations in $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$, three for each. Sections 4 and 5 contain proofs of the theorems. In the next two sections