Advances in Applied Mathematics 54 (2014) 1–10

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Permutation statistics of products of random permutations

Axel Hultman

Department of Mathematics, Linköping University, SE-581 83, Linköping, Sweden

ARTICLE INFO

Article history:
Received 8 August 2013
Received in revised form 30 October 2013
Accepted 30 October 2013
Available online 5 December 2013

MSC:
primary 60C05
secondary 05A05, 20B30

Keywords:
Symmetric group characters
Random walks
Permutation statistics

ABSTRACT

Given a permutation statistic $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$, define the *mean statistic* $\bar{\mathfrak{s}}$ as the class function giving the mean of $\mathfrak{s}$ over conjugacy classes. We describe a way to calculate the expected value of $\mathfrak{s}$ on a product of t independently chosen elements from the uniform distribution on a union of conjugacy classes $\Gamma \subseteq \mathfrak{S}_n$. In order to apply the formula, one needs to express the class function $\bar{\mathfrak{s}}$ as a linear combination of irreducible $\mathfrak{S}_n$-characters. We provide such expressions for several commonly studied permutation statistics, including the exceedance number, inversion number, descent number, major index and k-cycle number. In particular, this leads to formulae for the expected values of said statistics.

1. Introduction

Consider the symmetric group $\mathfrak{S}_n$ of permutations of $[n] = \{1, \dots, n\}$. For $\Gamma \subseteq \mathfrak{S}_n$ one may study the behaviour of various permutation statistics $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$ on products $\gamma_1 \cdots \gamma_t \in \mathfrak{S}_n$ of random $\gamma_i \in \Gamma$.

Definition 1.1. Choose a subset $\Gamma \subseteq \mathfrak{S}_n$, a function $\mathfrak{s} : \mathfrak{S}_n \to \mathbb{R}$ and a nonnegative integer t. We denote by $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ the expected value of $\mathfrak{s}$ on a product of t elements independently chosen from the uniform distribution on Γ.

A product of t random elements of Γ corresponds to a t-step random walk on the Cayley graph of $\mathfrak{S}_n$ induced by Γ. Random walks on Cayley graphs form a classical and well studied subject in probability theory; a good general reference is [13]. In the present paper, we specifically address the problem of computing $\mathbb{E}_\Gamma(\mathfrak{s}, t)$. Recent work in this vein includes the following. The case of Γ

E-mail address: axel.hultman@liu.se.

0196-8858/$ – see front matter
http://dx.doi.org/10.1016/j.aam.2013.10.003

being the set of adjacent transpositions and $\mathfrak{s}$ counting inversions was studied by Eriksson et al. [6], Eriksen [4] and Bousquet-Mélou [2]. Turning, instead, to Γ comprised of all transpositions, $\mathfrak{s}$ being the absolute length function (essentially counting disjoint cycles) was studied in [5], whereas Jönsson [10] considered the fixed point counting function $\mathfrak{s}$ and Sjöstrand [14] found the solution when $\mathfrak{s}$ counts inversions.

We shall describe a method to attack the general problem. Although it could potentially be of use for more general Γ (as indicated by the hypotheses of Theorem 3.1 below), we shall apply it to situations where Γ is a union of conjugacy classes. The technique makes use of representation theory of $\mathfrak{S}_n$. Similar ideas have been frequent in the study of random walks on Cayley graphs since the seminal paper by Diaconis and Shahshahani [3]. In particular, our method is heavily inspired by that described in [5]. We are concerned with more general Γ, but the principal novelty here is to dispose of the requisite of [5] that $\mathfrak{s}$ be a class function. As we shall see, removing this restriction significantly improves the versatility of the method. The aforementioned results from [5,10,14], as well as an abundance of new ones, are now accessible in a uniform way.

Here is a brief sketch of the content of the paper. In the next section, we review some facts about symmetric group characters. In Section 3, we then describe how they connect with the problem of computing $\mathbb{E}_\Gamma(\mathfrak{s}, t)$. The main observations of that section, Theorem 3.1 and Theorem 3.2, provide a recipe for calculating $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ whenever Γ is a union of conjugacy classes. In order to express $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ explicitly for a given statistic $\mathfrak{s}$, the remaining task is to decompose the *mean statistic* $\bar{\mathfrak{s}}$ as a linear combination of irreducible $\mathfrak{S}_n$-characters. This turns out to be a quite manageable task for many standard permutation statistics. We provide explicit decompositions for the mean statistics corresponding to the k-cycle number, exceedance number, weak exceedance number, inversion number, major index and descent number statistics in Sections 4, 5 and 6. In particular, $\mathbb{E}_\Gamma(\mathfrak{s}, t)$ is determined for $\mathfrak{s}$ being any of these statistics (and conjugation invariant Γ). We conclude with explicit examples in Section 7.

2. Symmetric group characters

In this section, we review elements of the representation theory of $\mathfrak{S}_n$. From this vast and classical subject only a few bits and pieces that we need in the sequel are extracted in order to agree on notation. We refer to [12] for a thorough background and much more information.

Let P_n denote the set of integer partitions $\lambda \vdash n$. The irreducible representations of $\mathfrak{S}_n$ are in bijection with P_n in a standard way. We use the notation ρ^λ for the representation indexed by $\lambda \vdash n$ and denote the corresponding character by χ^λ. These irreducible characters form a basis for the $\mathbb{C}$-vector space $\mathrm{Cl}_n = \{f : P_n \to \mathbb{C}\}$ of class functions. Moreover, this basis is orthonormal with respect to the standard Hermitian inner product on Cl_n

$$\langle f, g\rangle = \frac{1}{n!}\sum_{\lambda \vdash n} |C_\lambda| f(\lambda) g(\lambda)^*,$$

where C_λ denotes the conjugacy class of permutations π with $\mathrm{type}(\pi) = \lambda$ as cycle type.

Abusing notation, we at times consider class functions as defined on $\mathfrak{S}_n$ rather than on P_n. In other words, for $\pi \in \mathfrak{S}_n$ and $f \in \mathrm{Cl}_n$, $f(\pi)$ should be interpreted as $f(\mathrm{type}(\pi))$. We trust the context to prevent confusion.

It is convenient to encode partitions as weakly decreasing sequences of positive integers, sometimes employing exponent notation to signal repeated parts. For example, $(7, 3^4, 1)$ denotes the partition of 20 which consists of one part of size 7, four parts of size 3 and one part of size 1. In this notation, a *hook shape* is a partition of the form $(a, 1^b)$ for integers $a \geqslant 1$ and $b \geqslant 0$.

The trivial $\mathfrak{S}_n$-character is indexed by (n). Thus, $\chi^{(n)}(\mu) = 1$ for all $\mu \vdash n$. The next three lemmata collect a few more values of certain irreducible characters that we shall need in the sequel. All statements are readily verified using e.g. the Murnaghan–Nakayama rule.

Define $f^\lambda = \chi^\lambda((1^n))$. This is the dimension of the irreducible representation ρ^λ.