Advances in Applied Mathematics 54 (2014) 11–26

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

A circular order on edge-coloured trees and RNA m-diagrams ☆

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ARTICLE INFO

Article history:
Received 16 November 2012
Accepted 9 September 2013
Available online 15 December 2013

MSC:
primary 05C05, 05A15
secondary 37B10

Keywords:
Tree
Edge-coloured
Labelled triangulation
Fuss–Catalan number
Enumeration
Tree of relations
RNA secondary structure
Polygon
m-Angulation
Induction
Interval exchange transformation
RNA m-diagram

ABSTRACT

We study a circular order on labelled, m-edge-coloured trees with k vertices, and show that the set of such trees with a fixed circular order is in bijection with the set of RNA m-diagrams of degree k, combinatorial objects which can be regarded as RNA secondary structures of a certain kind. We enumerate these sets and show that the set of trees with a fixed circular order can be characterized as an equivalence class for the transitive closure of an operation which, in the case $m = 3$, arises as an induction in the context of interval exchange transformations.

☆ This work was supported by the Engineering and Physical Sciences Research Council [grant number EP/G007497/1], the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, and by the Leverhulme Trust in the form of an Early Career Fellowship for Sibylle Schroll.

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0196-8858/$ – see front matter
http://dx.doi.org/10.1016/j.aam.2013.09.009

1. Introduction

Interval exchange transformations are, roughly speaking, a generalization of a rotation of a circle. More precisely, given a partition of the unit interval into smaller segments, an interval exchange transformation is a map from the unit interval to itself where the segments are rearranged according to a choice of permutation. Recently there has been particular interest in the class of interval exchange transformations induced by the permutation $(k\ k-1\ \cdots\ 1)$. In [2] a combinatorial interpretation in terms of an induction process on labelled trees with edges coloured with 3 possible colours, called trees of relations, has proven a fruitful tool leading to new results in the understanding of the associated languages. In this paper we extend this combinatorics by defining a generalization of the induction process to the case of labelled trees with edges coloured with m possible colours and study its properties.

Given positive integers k and m, an *m-edge-coloured* tree with k vertices is a tree whose edges are labelled with m possible colours in such a way that no vertex is incident with two edges of the same colour. It is said to be *labelled* if its vertices are labelled with $\{1, 2, \ldots, k\}$. Such trees were shown to possess a circular order in the case $m = 3$ in [2] corresponding to the permutation of the interval exchange transformation. By giving a new proof of this fact, we show that the circular order can be generalized to arbitrary m and we study its relation to the generalized induction process we define.

Firstly, we show that the set $\mathcal{T}_{k,m}$ of labelled m-edge-coloured trees with k vertices and a fixed circular order is in bijection with a collection of combinatorial objects consisting of collections of noncrossing arcs in a disk with coloured marked points on its boundary, satisfying a certain matching rule. Such diagrams can be regarded as generalized RNA secondary structures of a certain kind, in the sense of [13], drawn in the Nussinov circle representation [10], so we refer to them as *RNA m-diagrams of degree k*. Such a bijection was given for the case $m = 3$ in [2].

We compute the cardinality of $\mathcal{T}_{k,m}$ (and thus also the number of RNA m-diagrams of degree k) using the well-known correspondence between trees and m-angulations of a polygon which induces a bijection with a collection of appropriately labelled m-angulations.

We prove the existence of a transformation of labelled m-edge-coloured trees preserving the circular order. We call this transformation *generalized induction* as it generalizes the induction in the case $m = 3$, studied combinatorially in [2]. In [4] it was shown that this induction (for $m = 3$) gives rise to languages naturally generalizing Sturmian languages and arising from interval exchange transformations. We also give an example showing that the most straightforward generalization of this induction to the case $m > 3$ does not in general preserve the circular order.

We go on to show that the equivalence classes of the transitive closure of the generalized induction are characterized by the circular order. Thus the set of equivalence classes is in bijection with the set of k-cycles in the symmetric group of degree k.

Finally, motivated by snake-triangulations and m-snakes in cluster theory, we give an interpretation of the generalized induction process in terms of labelled m-angulations of polygons.

This article is structured as follows. In Section 2, we introduce the combinatorial objects considered in the paper. In Section 3 we give the bijections mentioned above and compute the cardinality of the set of labelled m-edge-coloured trees with a fixed circular order. In Section 4 we show that the circular order characterizes the equivalence classes of the transitive closure of the generalized induction and in Section 5 we show that induction can be given by a composition of flips of diagonals in an m-angulation.

2. Some combinatorial objects

We first introduce the main combinatorial objects we will be considering: edge-coloured trees, RNA m-diagrams, and m-angulations.

Given positive integers k, m, we consider m-edge-coloured trees on k vertices, i.e. trees whose edges are coloured with one of the m symbols $S_1, S_2, \ldots, S_m$ in such a way that no two edges incident with the same vertex have the same colour. We say that such a tree is *labelled* if its vertices are labelled with $\{1, 2, \ldots, k\}$. As usual, if the connectedness assumption is not satisfied, we refer to the