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Minimum degree conditions for vertex-disjoint even cycles in large graphs ☆

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A B S T R A C T

We prove a variant of a theorem of Corrádi and Hajnal (1963) [4] which says that if a graph G has at least $3k$ vertices and its minimum degree is at least $2k$, then G contains k vertex-disjoint cycles. Specifically, our main result is the following. For any positive integer k, there is a constant c_k such that if G is a graph with at least c_k vertices and the minimum degree of G is at least $2k$, then (i) G contains k vertex-disjoint even cycles, or (ii) $(2k-1)K_1 \vee pK_2 \subset G \subset K_{2k-1} \vee pK_2$ $(p \geqslant k \geqslant 2)$, or (iii) $k = 1$ and each block of G is either a K_2 or an odd cycle.

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1. Introduction

In this paper, we consider only finite simple graphs. For terminology and notation not defined in this paper, we refer the readers to [5]. Let G be a graph. We denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ the vertex set, the edge set, the minimum degree and the maximum degree of G, respectively. We refer to the cardinality of $V(G)$ as the *order* of G and denote it by $|G|$. For a graph H, if H is a subgraph of G, then we write $H \subset G$. Subgraphs of G are said to be *vertex-disjoint* if no two of them have any common vertex in G. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X, and let $G - X = G[V(G) \setminus X]$. For $H \subset G$, let $G - H = G - V(H)$.

Packing and covering is one of the central areas in both graph theory and theoretical computer science. The starting point of this research area goes back to the following well-known theorem due to Erdős and Pósa [7] in early 1960s.

Theorem A. *(See Erdős and Pósa [7].) For any integer k with $k \geqslant 1$ and any graph G, either G contains k vertex-disjoint cycles or a vertex set X of order at most $f(k)$ (for some function f of k) such that $G - X$ is a forest.*

In fact, Theorem A gives rise to the well-known Erdős–Pósa property. A family $\mathcal{F}$ of graphs is said to have the *Erdős–Pósa property*, if for every integer $k \geqslant 1$, there is an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set X of at most $f(k, \mathcal{F})$ vertices such that $G - X$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term Erdős–Pósa property arose because of Theorem A which proves that the family of cycles has this property.

Theorem A concerns both "packing", i.e., k vertex-disjoint cycles, and "covering", i.e., a set of at most $f(k)$ vertices that hits all the cycles in G. Starting with this result, there are a lot of the results in this direction. Packing appears almost everywhere in extremal graph theory, while covering leads to the well-known concept "feedback set" in theoretical computer science. Also, the cycle packing problem, which asks whether or not there are k vertex-disjoint cycles in an input graph G, is a well-known problem, e.g. [12].

In graph theory, there are many results concerning packing cycles. The following is the well-known theorem due to Corrádi and Hajnal [4] in 1960s.

Theorem B. *(See Corrádi and Hajnal [4].) Let k be an integer with $k \geqslant 1$, and let G be a graph of order at least $3k$. If $\delta(G) \geqslant 2k$, then G contains k vertex-disjoint cycles.*

Theorem B tells us that if we assumed that the minimum degree of a given graph is at least $2k$, then the covering result in Theorem A would not happen. In view of Theorems A and B, we would like to discuss how a parity condition on the cycles affects Theorems A

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