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On gamma quotients and infinite products

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ABSTRACT

Convergent infinite products, indexed by all natural numbers, in which each factor is a rational function of the index, can always be evaluated in terms of finite products of gamma functions. This goes back to Euler. A purpose of this note is to demonstrate the usefulness of this fact through a number of diverse applications involving multiplicative partitions, entries in Ramanujan's notebooks, the Chowla–Selberg formula, and the Thue–Morse sequence. In addition, we propose a numerical method for efficiently evaluating more general infinite series such as the slowly convergent Kepler–Bouwkamp constant.

1. Introduction

Recall that an infinite product $\prod_{k=1}^{\infty} a(k)$ is said to converge if the sequence of its partial products converges to a nonzero limit. In this note we are especially interested in the case when $a(k)$ is a rational function of k. Assuming that the infinite product converges, $a(k)$ is then necessarily of the form

$$a(k) = \frac{(k+\alpha_1)\cdots(k+\alpha_n)}{(k+\beta_1)\cdots(k+\beta_n)} \tag{1}$$

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with $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ complex numbers, none of which are negative integers, such that $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$. To see that this is the case, note first that clearly $a(k) \to 1$ as $k \to \infty$ so that $a(k)$ can be factored into linear terms as on the right-hand side of (1). On the other hand, if $a(k) = 1 + ck^{-1} + O(k^{-2})$ as $k \to \infty$ then convergence of the infinite product (and divergence of the harmonic series) forces $c = \alpha_1 + \cdots + \alpha_n - \beta_1 - \cdots - \beta_n = 0$.

These infinite products always have a finite term evaluation in terms of Euler's gamma function [30, Section 12.13].

Theorem 1.1. *Let $n \geqslant 1$ be an integer, and let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ be nonzero complex numbers, none of which are negative integers. If $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$, then*

$$\prod_{k \geqslant 0} \frac{(k+\alpha_1)\cdots(k+\alpha_n)}{(k+\beta_1)\cdots(k+\beta_n)} = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}. \tag{2}$$

Otherwise, the infinite product in (2) diverges.

This result is a simple consequence of Euler's infinite product definition (3) of the gamma function, see the beginning of Section 2. It is, however, scarcely stated explicitly in the literature. For instance, while the table [14] contains several pages of special cases of (2), some of which are rather generic in nature, it does not list (2) or an equivalent version thereof. An incidental objective of this note is therefore to advertise (2) and to illustrate its usefulness during the course of the applications given herein.

This note was motivated by a result, discussed in Section 3, which recently appeared in [8] as part of a study of multiplicative partitions. In Section 3 we also apply Theorem 1.1 to two entries in Ramanujan's (lost) notebook [3].

The first novel contribution of this note may be found in Section 4, where we propose an approach to the numerical evaluation of certain general, not necessarily rational, infinite products, which is based upon Theorem 1.1 and Padé approximation. We illustrate this approach by applying it to the Kepler–Bouwkamp constant, defined as the infinite product $\prod_{k=3}^{\infty} \cos(\pi/k)$. Due to its infamously slow convergence, various procedures for its numerical evaluation have been discussed in the literature [7,13,28]. The present approach has the advantage that it does not rely on developing alternative, more rapidly convergent, expressions for the Kepler–Bouwkamp constant.

In Section 5 we discuss properties of short gamma quotients at rational arguments. In particular, we offer an alternative proof of a result established in [24] and [17]. In the light of our proof, this result may be interpreted as a (much simpler) version of the Chowla–Selberg formula [25] in the case of principal characters.

Finally, in Section 6, consideration of an infinite product defined in terms of the Thue–Morse sequence naturally leads us to a curious open problem posed by Shallit.

2. Proof and basic examples

We commence with supplying a proof of Theorem 1.1 and giving a number of basic examples.

Proof of Theorem 1.1. Euler's definition gives the gamma function as

$$\Gamma(z) = \lim_{m\to\infty} \frac{m^z m!}{z(z+1)\cdots(z+m)}, \tag{3}$$

which is valid for all $z \in \mathbb{C}$ except for negative integers z. Thus,

$$\prod_{j=1}^{n} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)} = \lim_{m\to\infty} \prod_{j=1}^{n} m^{\beta_j-\alpha_j} \prod_{k=0}^{m} \frac{\alpha_j+k}{\beta_j+k} = \lim_{m\to\infty} \prod_{k=0}^{m} \prod_{j=1}^{n} \frac{\alpha_j+k}{\beta_j+k},$$