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A generalization of carries processes and Eulerian numbers

Fumihiko Nakano [a], Taizo Sadahiro [b,*]

[a] *Department of Mathematics, Gakushuin University, 1-5-1, Mejiro, Toshima-ku, Tokyo, 171-8588, Japan*
[b] *Department of Computer Science, Tsuda College, Tokyo, Japan*

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ABSTRACT

We study a generalization of Holte's amazing matrix, the transition probability matrix of the Markov chains of the 'carries' in a *non-standard* numeration system. The stationary distributions are explicitly described by the numbers which can be regarded as a generalization of the Eulerian numbers and the MacMahon numbers. We also show that similar properties hold even for the numeration systems with the negative bases.

1. Introduction and statements of results

The transition probability matrix so-called 'amazing matrix' of the Markov chain of the 'carries' has very nice properties [5], and has unexpected connection to the Markov chains of riffle shuffles [2, 3]. Diaconis and Fulman [3] studies a variant of the carries process, type B carries process. Novelli and Thibon studies the carries process in terms of noncommutative symmetric functions [7]. This paper studies a generalization of the carries process which includes Diaconis and Fulman's type B carries process as a special case. We study the transition probability matrices of the Markov chains of the carries in the numeration systems with non-standard digit sets. We show that the matrices have the eigenvectors which can be perfectly described by a generalization of Eulerian numbers and the MacMahon numbers [8,6,1,3]. We also show that similar properties hold even for the numeration systems with negative bases.

* Corresponding author.
E-mail addresses: fumihiko@math.gakushuin.ac.jp (F. Nakano), sadahiro@tsuda.ac.jp (T. Sadahiro).

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$$
\begin{array}{ccccccc}
 & \cdots & C_4 & C_3 & C_2 & C_1 & C_0 \\
 & & & & & & \\
 & \cdots & X_{1,4} & X_{1,3} & X_{1,2} & X_{1,1} & X_{1,0} \\
 & \cdots & X_{2,4} & X_{2,3} & X_{2,2} & X_{2,1} & X_{2,0} \\
 & \cdots & X_{3,4} & X_{3,3} & X_{3,2} & X_{3,1} & X_{3,0} \\
 & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
+) & \cdots & X_{n,4} & X_{n,3} & X_{n,2} & X_{n,1} & X_{n,0} \\
\hline
 & \cdots & A_4 & A_3 & A_2 & A_1 & A_0
\end{array}
$$

Fig. 1. Carries process.

1.1. Numeration system

Throughout the paper, b denotes a positive integer and $\mathcal{D} = \{d, d+1, \ldots, d+b-1\}$ denotes a set of integers containing 0. Therefore, $-b < d < b$. Then, we have a numeration system $(b, \mathcal{D})$: Suppose that an integer x has a representation of the form,

$$x = (x_k x_{k-1} \cdots x_0)_b \overset{def}{=} x_0 + x_1 b + x_2 b^2 + \cdots + x_k b^k, \quad x_0, x_1, \ldots, x_k \in \mathcal{D},\ x_k \neq 0. \tag{1}$$

Then, it can be easily shown that this representation is uniquely determined for x and

$$\left\{(x_k x_{k-1} \cdots x_0)_b \;\middle|\; k \geqslant 0, x_0, x_1, \ldots, x_k \in \mathcal{D}\right\} = \begin{cases} \mathbb{Z} & d \neq 0, -b+1, \\ \mathbb{N} & d = 0, \\ -\mathbb{N} & d = -b+1 \end{cases}$$

is closed under the addition, where $\mathbb{N}$ denotes the set of non-negative integers.

1.2. Carries process

Let $\{X_{i,j}\}_{1 \leqslant i \leqslant n, j \geqslant 0}$ be the set of independent random variables each of which is distributed uniformly over $\mathcal{D}$. Define the two stochastic processes $(A_0, A_1, A_2, \ldots)$ and $(C_0, C_1, C_2, \ldots)$ in the following way: $C_0 = 0$ with probability one. $(A_i)_{i \geqslant 0}$ is a sequence of $\mathcal{D}$-valued random variables satisfying

$$A_i \equiv C_i + X_{1,i} + \cdots + X_{n,i} \pmod b, \quad i = 0, 1, 2, \ldots,$$

and

$$C_i = \frac{C_{i-1} + X_{1,i-1} + \cdots + X_{n,i-1} - A_{i-1}}{b}, \quad i = 1, 2, 3, \ldots.$$

(See Fig. 1.) It is obvious that $(C_0, C_1, C_2, \ldots)$ is a Markov process, which we call the *carries process* with n summands or simply *n-carry process* over $(b, \mathcal{D})$.

1.3. A generalization of Eulerian numbers

Let $p \geqslant 1$ be a real number and n a positive integer. Then we define an array of numbers $v_{i,j}^{(p)}(n)$ for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, n+1$ by

$$v_{i,j}^{(p)}(n) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \bigl[p(j-r)+1\bigr]^{n-i}, \tag{2}$$