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Difference integrability conditions for parameterized linear difference and differential equations

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ABSTRACT

This paper is devoted to integrability conditions for systems of linear difference and differential equations with difference parameters. It is shown that such a system is difference isomonodromic if and only if it is difference isomonodromic with respect to each parameter separately. Due to this result, it is no longer necessary to solve non-linear difference equations to verify isomonodromicity, which will improve efficiency of computation with these systems.

1. Introduction

In this paper, we improve the algorithm that verifies if a system of linear difference and differential equations with difference parameters is isomonodromic (Definition 3.1). Given a system of difference or differential equations with parameters, it is natural to ask if its solutions satisfy extra linear difference equations with respect to the parameters. Consider an algorithm whose input consists of a field K with commuting automorphisms $\phi_1, \ldots, \phi_q, \sigma_1, \ldots, \sigma_r$ and derivations $\partial_1, \ldots, \partial_m$ on K and a system of difference equations

$$\phi_1(Y) = A_1 Y, \ldots, \phi_q(Y) = A_q Y, \qquad \partial_1 Y = B_1 Y, \ldots, \partial_m Y = B_m Y, \tag{1}$$

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where $A_i \in \mathbf{GL}_n(K)$ and $B_j \in \mathbf{M}_n(K)$, $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant m$. Its output consists of such additional linear difference equations

$$\sigma_1 Y = C_1 Y, \quad \ldots, \quad \sigma_r Y = C_r Y, \tag{2}$$

where $C_i \in \mathbf{GL}_n(K)$, $1 \leqslant i \leqslant r$, if they exist, for which there exists an invertible matrix solution of (1) that satisfies (2) as well. For an invertible matrix solution of (1) to possibly exist, the A_i's and B_j's must satisfy the integrability conditions (5), (6), and (7). Moreover, the existence of the C_i's above is equivalent to the existence of C_i's satisfying another large collection of integrability conditions (9), (10) and (11). Such systems (1) are called isomonodromic in analogy with differential equations [26,27,19,20,16]. Isomonodromy problems for q-difference equations and their relations with q-difference Painlevé equations were studied in [21,22].

The main result, Theorem 4.1, states that (11), which are non-linear difference equations in the C_i's, do not have to be verified to check the existence of the C_i's. More precisely, the existence of C_i's satisfying (9) and (10) implies the existence of new invertible matrices that satisfy all of (9), (10), and (11).

Since, due to our result, we only need to check the existence of solutions (that have entries in the ground field K) of a system of linear difference and differential equations, a complexity estimate for verifying whether a system of difference and differential equations is isomonodromic becomes possible due to [1,2,4,7,6,8].

A similar problem but for systems of differential equations was considered in [16], motivated by the classical results [19,20]. Differential categories developed in [15] formed the main technical tool in [16]. On the contrary, our proofs are constructive, which makes them more suitable for practical use. In particular, from our proof, one can derive an algorithm that, given a common invertible solution of (9) and (10), computes a common invertible solution of all (9), (10), and (11).

If one attempts to find a common solution for all of (9), (10), and (11) in a naive way, one might have to adjoin new elements to the base field K that are not constant with respect to the derivations and automorphisms, which is not desirable. On the other hand, Theorem 4.1 shows how one can limit, in a computable way, these newly adjoint elements to those that are constant with respect to all derivations and the automorphisms that do not correspond to the parameters, which is another advantage of this approach. This is also different from the approach taken in [5] to give an explicit treatment of a similar problem but for differential parameters, where a linearly differentially closed assumption is imposed on the field of constants with respect to the automorphisms.

The main result is expected to have further applications. In parameterized differential Galois theory [9], there are several algorithms for computing Galois groups. For 2×2 systems, they are given in [3,14]. An algorithm, more general in terms of the order of the system [24,23] uses the differential analogues [16] of our results to make the computation more efficient. Our results may prove useful in the design of an algorithm for computing parameterized difference Galois groups with difference parameters, as it has happened in the differential case, once [13] or [25] is generalized to the case of several difference parameters, which is a challenge on its own.

These and other difference and differential Galois theories [18,17,11,12,10,28] have been developed to compute all difference and differential algebraic relations that solutions of systems of linear difference and differential equations satisfy and have found many practical applications.

The paper is organized as follows. Notation is introduced and the basic notions of differential and difference algebra are reviewed in Section 2. The integrability conditions are discussed in Section 3, where the main result is illustrated by showing a concrete example, in which we perform computation with the basic hypergeometric q-difference equation with parameters (Example 3.2). The main result and its proof are described in Section 4.

2. Basic definitions

A Δ-ring is a commutative associative ring with unit 1 together with a set $\Delta = \{\partial_1, \ldots, \partial_m\}$ of commuting derivations $\partial_i : R \to R$ such that

$$\partial_i(a+b) = \partial_i(a) + \partial_i(b), \qquad \partial_i(ab) = \partial_i(a)b + a\partial_i(b), \quad a, b \in R,\ 1 \leqslant i \leqslant m.$$