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Using Noonan–Zeilberger Functional Equations to enumerate (in polynomial time!) generalized Wilf classes ☆

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ABSTRACT

One of the most challenging problems in enumerative combinatorics is to count Wilf classes, where you are given a pattern, or set of patterns, and you are asked to find a "formula", or at least an efficient algorithm, that inputs a positive integer n and outputs the number of permutations avoiding that pattern. In 1996, John Noonan and Doron Zeilberger initiated the counting of permutations that have a prescribed, r, say, occurrences of a given pattern. They gave an ingenious method to generate functional equations, alas, with an unbounded number of "catalytic variables", but then described a clever way, using multivariable calculus, on how to get enumeration schemes. Alas, their method becomes very complicated for r larger than 1. In the present article we describe a far simpler way to squeeze the necessary information, in polynomial time, for increasing patterns of any length and for any number of occurrences r.

1. Introduction

Recall that the *reduction* of a finite list of k, say, distinct (real) numbers $[a_1, a_2, \dots, a_k]$ is the unique permutation $\sigma = [\sigma_1, \dots, \sigma_k]$, of $\{1, \dots, k\}$ such that a_1 is the σ_1-th largest element in the list, a_2 is the σ_2-th largest element in the list, etc. In other words $[a_1, a_2, \dots, a_k]$ and σ are "order-isomorphic". For example, the reduction of $[6, 3, 8, 2]$ is $[3, 2, 4, 1]$ and the reduction of $[\pi, \gamma, e, \phi]$ is $[4, 1, 3, 2]$ (where ϕ is the Golden Ratio).

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Given a permutation $\pi = \pi_1 \dots \pi_n$ and another permutation $\sigma = [\sigma_1, \dots, \sigma_k]$ (called a *pattern*), we denote by $N_\sigma(\pi)$ the number of instances $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that the reduction of $\pi_{i_1} \dots \pi_{i_k}$ is σ.

For example, if $\pi = 51324$ then

$N_{[1,2,3]}(\pi) = 2$ (because $\pi_2\pi_3\pi_5 = 134$ and $\pi_2\pi_4\pi_5 = 124$ reduce to $[1, 2, 3]$).
$N_{[1,3,2]}(\pi) = 1$ (because $\pi_2\pi_3\pi_4 = 132$ reduces to $[1, 3, 2]$).
$N_{[2,1,3]}(\pi) = 1$ (because $\pi_3\pi_4\pi_5 = 324$ reduces to $[2, 1, 3]$).
$N_{[2,3,1]}(\pi) = 0$ (because none of the 10 length-three subsequences of π reduces to 231).
$N_{[3,1,2]}(\pi) = 5$ (because $\pi_1\pi_2\pi_3 = 513$ and $\pi_1\pi_2\pi_4 = 512$ and $\pi_1\pi_2\pi_5 = 514$ and $\pi_1\pi_3\pi_5 = 534$ and $\pi_1\pi_4\pi_5 = 524$ all reduce to $[3, 1, 2]$) .
$N_{[3,2,1]}(\pi) = 1$ (because $\pi_1\pi_3\pi_4 = 532$ reduces to $[3, 2, 1]$).

Of course the sum of $N_\sigma(\pi)$ over all k-permutations σ is $\binom{n}{k}$.

Fixing a pattern σ, the set of permutations π for which $N_\sigma(\pi) = 0$ (we say that π *avoids* σ) is called the *Wilf class* of σ, and more generally, given a set of patterns S, the set of permutations for which $N_\sigma(\pi) = 0$ for all $\sigma \in S$, is the Wilf class of that set. The first *systematic* study of *enumerating* Wilf classes was undertaken in the pioneering paper by Rodica Simion and Frank Schmidt [13].

The general question is extremely difficult (see [14] and [3]) and "explicit" answers are only known for few short patterns (and sets of patterns), the increasing patterns $[1, 2, \dots, k]$, and a few other *West-equivalent* to them, giving the same enumeration. For example, even for the pattern $[1, 3, 2, 4]$ (`http://oeis.org/A061552`) the best known algorithm takes exponential time in n, and it is very possible that that's the best that one can do.

But for those patterns σ for which we know how to enumerate their Wilf classes, most importantly the increasing patterns $[1, \dots, k]$, it makes sense to ask the more general question:

Given a pattern σ and a positive integer r, find a "formula", or at least a polynomial time algorithm (thus *answering* the question in the sense of Herb Wilf [15]) that inputs a positive integer n and outputs the number of permutations π of $\{1, \dots, n\}$ for which $N_\sigma(\pi) = r$. We call such a class a *generalized* Wilf class.

Ideally, we would like to have, given a pattern σ, an explicit formula, in n and q, for the generating function (S_n denotes the set of permutations of $\{1, \dots, n\}$)

$$A_\sigma(q, n) := \sum_{\pi \in S_n} q^{N_\sigma(\pi)},$$

then, for any fixed r, the sequence of coefficients of q^r in $A_\sigma(q, n)$ would give the sequence enumerating permutations with *exactly* r occurrences of the pattern σ.

In fact, for patterns of length $\leqslant 2$ there are nice answers. Trivially

$$A_{[1]}(q, n) := n!q^n,$$

and almost-trivially (or at least classically)

$$A_{[2,1]}(q, n) := (1)(1 + q) \dots \left(1 + q + \cdots + q^{n-1}\right) = [n]!,$$

the famous "q-analog" of $n!$. But things start to get complicated for patterns of length 3.

2. Past work

For a very lucid and extremely engaging introduction to the subject, as well as the state-of-the-art, we strongly recommend Miklós Bóna's *masterpiece* [3].