Advances in Applied Mathematics 50 (2013) 390–408

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Discrete Tomography and plane partitions

P. Dulio [a,*], C. Peri [b]

[a] Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy
[b] Università Cattolica S. C., Via Emilia Parmense 84, I-29122 Piacenza, Italy

ARTICLE INFO

Article history:
Received 25 January 2012
Accepted 12 September 2012
Available online 26 October 2012

MSC:
primary 05A17, 05D05
secondary 11P81, 15A36

Keywords:
Additivity
Bad-configuration
Plane partition
Pyramid
Reconstruction
Uniqueness

ABSTRACT

A *plane partition* is a $p \times q$ matrix $A = (a_{ij})$, where $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$, with non-negative integer entries, and whose rows and columns are weakly decreasing. From a geometric point of view plane partitions are equivalent to *pyramids*, subsets of the integer lattice $\mathbb{Z}^3$ which play an important role in Discrete Tomography. As a consequence, some typical problems concerning the tomography of discrete lattice sets can be rephrased and considered via plane partitions. In this paper we focus on some of them. In particular, we get a necessary and sufficient condition for additivity, a canonical procedure for checking the existence of (weakly) bad configurations, and an algorithm which constructs minimal pyramids (with respect to the number of levels) with assigned projection of a bad configurations.

1. Introduction

In several papers the notion of plane partition has been considered, since the early works of P.A. MacMahon (who collected his results in this area in [4, Sections IX and X]). More recently plane partitions have been reinterpreted in terms of pyramids (see, for instance [7–11]). This provides an important link with Discrete Tomography and its typical problems, where sets of uniqueness and additive sets are frequently looked for. (See Section 2 for all terminology.) It is known that every additive set is a set of uniqueness, but not conversely [1, Theorems 2 and 5] or [2, Section 2.5]. From this point of view pyramids play a special role, since a necessary and sufficient condition for a pyramid to be additive is known by [9, Theorem 1]. Also, assuming that the slice vectors are weakly decreasing, every set of uniqueness is a pyramid [9, Corollary 3.2]. In [1], with a particular combinatorial argument, an example of pyramid which is a non-additive set of uniqueness is exhibited. In [8, Example 3.7],

* Corresponding author.
E-mail addresses: paolo.dulio@polimi.it (P. Dulio), carla.peri@unicatt.it (C. Peri).

0196-8858/$ – see front matter
http://dx.doi.org/10.1016/j.aam.2012.10.005

a *thickening* of such a pyramid proves to have a bad configuration, and consequently it is not a set of uniqueness. On the other hand in [9], by an algebraic approach, a slightly modified construction is shown to be an additive pyramid. These results suggest the following natural question: how are such different behaviors of quite similar sets related to their representations by plane partitions?

As an attempt to answer this question, we address the following problems.

Problem 1. Find a characterization of additive pyramids uniquely based on the entries of the associated plane partition.

Problem 2. Find a canonical procedure for checking the existence of a (weakly) bad configuration in a given pyramid.

Problem 3. Let $F = (Z_0, W_0) \subset \mathbb{Z}^2$ be a switching component with respect to the coordinate directions, and let $\pi : \mathbb{Z}^3 \to \mathbb{Z}^2$ be the orthogonal projection $\pi(x_1, x_2, x_3) = (x_1, x_2, 0)$. (Here $\mathbb{Z}^2$ is identified with the set of points $(x_1, x_2, 0) \in \mathbb{Z}^3$.) Is there an algorithm which constructs a minimal (with respect to the number of levels) non-additive pyramid, with a (weakly) k-bad configuration (Z^*, W^*) such that $\pi(Z^*) = Z_0$ and $\pi(W^*) = W_0$?

Motivations to Problem 1 come from [9, Theorem 1], where the additivity of a given pyramid can be easily checked by means of two suitable lists of real numbers (see Theorem 1). This is a very interesting result, though the construction of such lists of numbers is not explicitly given, as well as their geometric meaning is not apparent. Therefore, an intrinsic condition (depending only on the associated plane partition) should be desirable.

Problem 2 raises from the analysis of some non-additive pyramids that we know from the literature. We found that the non-additivity property is provided by exhibiting (weakly) bad configurations which come out with no explicit construction. Our purpose is to show a general procedure which finds possible (weakly) bad configuration in a given pyramid.

It can be easily shown, via the orthogonal projection $\pi : \mathbb{Z}^3 \to \mathbb{Z}^2$, that non-additive pyramids provide a link between two important tomographic notions, namely zero-one normalized maximal matrices (see [5]), and switching-components with respect to the coordinate directions (see [3] for an algebraic characterization of switching-components). More precisely, the orthogonal projection of a pyramid S determines a set of lattice points in $\mathbb{Z}^2$ corresponding to a maximal zero-one matrix, simply obtained by normalizing the entries of the plane partition associated to S. Further, the orthogonal projection of a (weakly) bad configuration $(W, Z) \subset \mathbb{Z}^3$ for S (with respect to the coordinate planes) determines a switching-component $(Z_0, W_0) \subset \mathbb{Z}^2$ (with respect to the coordinate directions). The inverse problem of reconstructing non-additive pyramids, and hence the corresponding plane partition, from the knowledge of (Z_0, W_0) sounds tomographically interesting, also in view of a better understanding of ambiguous (non-unique) solutions. Problem 3 addresses this question under a constraint of minimality with respect to the number of levels.

The paper is organized as follows. In Section 2 the main notations and definitions are supplied, together with some preliminary results. In Section 3 we give a necessary and sufficient condition for a pyramid to be non-additive (Theorem 2) which gives an answer to Problem 1. As a consequence we get Corollary 4, which represents an extension to k-bad configurations ($k > 2$) of a result concerning 3-bad configurations obtained by E. Vallejo in [8] (it is explicitly derived as a particular case of Corollary 4 in Appendix A). In Section 4 the notion of *corner point* is introduced, and an answer to Problem 2 is provided in Theorem 8. In Section 5 we provide an explicit construction which solves Problem 3, together with a few examples.

2. Notations and preliminary results

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural and integer numbers, respectively. For $m \in \mathbb{N}$ we denote by $[m]$ the set $\{1, \ldots, m\}$. Also, for $p, q, r \in \mathbb{N}$, let $B(p, q, r) = [p] \times [q] \times [r]$ denote the three-dimensional box in the integer lattice $\mathbb{Z}^3$, with sides parallel to the coordinate axes. For any subset