Advances in Applied Mathematics 50 (2013) 445–464

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

A flow approach to the L_{-2} Minkowski problem

Mohammad N. Ivaki

Department of Mathematics and Statistics, Concordia University, Montreal, QC, H3G 1M8, Canada

A R T I C L E I N F O

Article history:
Received 4 May 2012
Accepted 10 September 2012
Available online 17 November 2012

MSC:
primary 53C44, 53A04, 52A10, 53A15
secondary 35K55

Keywords:
L_{-2} Minkowski problem
Support function
p-Affine surface area
Affine support function
Affine arc-length

A B S T R A C T

We prove that the set of smooth, π-periodic, positive functions on the unit sphere for which the planar L_{-2} Minkowski problem is solvable is dense in the set of all smooth, π-periodic, positive functions on the unit sphere with respect to the L^∞ norm. Furthermore, we obtain a necessary condition on the solvability of the even L_{-2} Minkowski problem. At the end, we prove uniqueness of the solutions up to special linear transformations.

1. Introduction

In differential geometry, the classical Minkowski problem concerns the existence, uniqueness and regularity of closed convex hypersurfaces whose Gauss curvature is prescribed as function of the normals. More generally, the Minkowski problem asks what are the necessary and sufficient conditions on a Borel measure on $\mathbb{S}^{n-1}$ to guarantee that it is the surface area measure of a convex body in $\mathbb{R}^n$. If the measure μ has a smooth density Φ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^{n-1}$, the Minkowski problem is equivalent to the study of solutions to the following partial differential equation on the unit sphere

$$\det(\bar{\nabla}^2 s + \mathrm{Id}\, s) = \Phi,$$

where $\bar{\nabla}$ is the covariant derivative on $\mathbb{S}^{n-1}$ endowed with an orthonormal frame. Note that for a smooth convex body K with support function s, the quantity $\det(\bar{\nabla}^2 s + \mathrm{Id}\, s)$ is the reciprocal of the

E-mail address: mivaki@mathstat.concordia.ca.

0196-8858/$ – see front matter
http://dx.doi.org/10.1016/j.aam.2012.09.003

Gauss curvature of the boundary of K. The answer to the existence and uniqueness of the Minkowski problem is as follows. If the support of μ is not contained in a great subsphere of $\mathbb{S}^{n-1}$, and it satisfies

$$\int_{\mathbb{S}^{n-1}} z\, d\mu(z) = 0,$$

then it is the surface area of a convex body, and the solution is unique up to a translation. Minkowski himself solved the problem in the category of polyhedrons. A.D. Alexandrov and others solved the problem in general, however, without any information about the regularity of the (unique) convex hypersurface. Around 1953, L. Nirenberg (in dimension three) and A.V. Pogorelov (in all dimensions) solved the regularity problem in the smooth category independently. For references, one can see works by Minkowski [38,39], Alexandrov [2–4], Fenchel and Jessen [19], Lewy [28,29], Nirenberg [40], Calabi [13], Cheng and Yau [15], Caffarelli et al. [12], and others.

In Lutwak's development of Brunn–Minkowski–Firey theory [30,31], it has been shown that the Minkowski problem is part of a larger family of problems called the L_p Minkowski problems. In the L_p Brunn–Minkowski–Firey theory, Lutwak introduced the notion of the L_p surface area. Therefore, it is natural to ask what are the necessary and sufficient conditions on a Borel measure on $\mathbb{S}^{n-1}$ which guarantee that it is the L_p surface area measure of a convex body. For $p \geqslant 1$, and an even measure, existence and uniqueness of the convex body was established by Lutwak [30]. If the measure μ has a smooth density Φ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^{n-1}$, the L_p problem is equivalent to the study of solutions to the following Monge–Ampère equation on the unit sphere

$$s^{1-p} \det\left(\bar{\nabla}^2 s + \mathrm{Id}\, s\right) = \Phi,$$

where $\bar{\nabla}$ is the covariant derivative on $\mathbb{S}^{n-1}$ endowed with an orthonormal frame. Note that for $p = 1$ this is the classical Minkowski problem. Solutions to many cases of these generalized problems followed later by Ai, Chou, Andrews, Böröczky, Chen, Wang, Gage, Guan, Lin, Jiang, Lutwak, Oliker, Yang, Zhang, Stancu, Umanskiy [1,7,9,10,14,16,20–22,25,32,33,36,42–44,47]. The progress in studying L_p Minkowski problems has been extremely fruitful and resulted in many applications to functional inequalities [17,33,36,37,34,35]. This unified theory relates many problems that, previously, were not connected. Note also that, for constant data Φ, many L_p problems were treated as self-similar solutions of geometric flows [5–7,9,20,21] and others.

The cases $p = -n$ and $p = 0$ are quite special and more difficult. The even case $p = 0$ has been recently solved by Böröczky, Lutwak, Yang and Zhang [10]. Many challenges remain for the problem with $p < 1$ and, particularly, for negative p. The above partial differential equation with $p \in [-2, 0]$ and $n = 2$ has been studied by Chen [14] and more recently by Jiang [25] for Φ not necessarily positive. For $p \leqslant -2$, some existence results were obtained by Dou and Zhu including generalizing the result obtained by Jiang in the case $p = -2$ [18].

The smooth L_{-n} Minkowski problem is technically more complex, than the well-known counterpart, the Minkowski problem in the classical differential geometry. It is the problem which seeks necessary and sufficient conditions for the existence of a solution to a particular affine invariant, fully nonlinear partial differential equation. It is essential to say, the term *centro* in centro-affine differential geometry emphasizes that, contrary to affine differential geometry or classical differential geometry, Euclidean translations of an object in the ambient space are not allowed. This generates a bothersome obstacle for studying the L_{-n} problem in full generality. Previous investigations of the L_{-n} Minkowski problem have been restricted to the even L_{-n} Minkowski problem, e.g., the problem in which it is assumed μ has the same values on antipodal Borel sets [1,14,16,25].

Let K be a compact, centrally symmetric, strictly convex body, smoothly embedded in $\mathbb{R}^2$. We denote the space of such convex bodies by $\mathcal{K}_{sym}$. Let

$$x_K : \mathbb{S}^1 \to \mathbb{R}^2$$