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Some improvements of the S-adic conjecture

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ABSTRACT

In [S. Ferenczi, Rank and symbolic complexity, Ergodic Theory Dynam. Systems 16 (1996) 663–682], S. Ferenczi proved that the language of any uniformly recurrent sequence with an at most linear complexity is S-adic. In this paper we adapt his proof in a more structured way and improve this result stating that any such sequence is itself S-adic. We also give some properties on the constructed morphisms.

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1. Introduction

A usual tool in the study of sequences (or infinite words) over a finite alphabet A is the complexity function p that counts the number of factors of each length n occurring in the sequence. This function is clearly bounded by d^n, $n \in \mathbb{N}$, where d is the number of letters in A but not all functions bounded by d^n are complexity functions. As an example, it is well known (see [19]) that either the sequence is ultimately periodic (and then $p(n)$ is ultimately constant), or its complexity function grows at least like $n+1$. Non-periodic sequences with minimal complexity $p(n) = n+1$ for all n exist and are called *Sturmian sequences* (see [19]). These words are binary sequences (because $p(1) = 2$) and admit several equivalent definitions: aperiodic balanced sequences, codings of rotations, mechanical words of irrational slope, etc. See Chapter 2 of [18] and Chapter 6 of [17] for surveys on these sequences. In particular, it is well known that all these sequences can be generated with only three morphisms.

Many other known sequences have a low complexity. By "low complexity" we usually mean "complexity bounded by a linear or affine function". Fixed points of primitive substitutions, automatic sequences, linearly recurrent sequences (see [12]) and Arnoux–Rauzy sequences are exam-

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ples of sequences with an at most affine complexity. For any such sequence $\mathbf{w}$, there exists a finite set S of morphisms over an alphabet A, a letter a and a sequence $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ such that $\mathbf{w} = \lim_{n\to\infty} \sigma_0 \cdots \sigma_n(a^\omega)$. Indeed, automatic sequences can be seen as images under letter-to-letter morphisms of fixed points of uniform substitutions (see [1]), F. Durand proved it in [10] and [11] for linearly recurrent sequences and P. Arnoux and G. Rauzy proved it in [2] for the so-called Arnoux–Rauzy sequences. Following [17], a sequence $\mathbf{w}$ with previous property is said to be S-adic (where S refers to the set of morphisms).

As mentioned in [17], the S-adic conjecture is the existence of a condition C such that "*a sequence has an at most linear complexity if and only if it is an S-adic sequence verifying C*". It is not possible to avoid considering a particular condition since, for instance, there exist fixed points of morphisms with a quadratic complexity (see [20]) and moreover, J. Cassaigne recently showed that there exists a finite set S of morphisms over an alphabet $A \cup \{l\}$ (where l is a special letter that does not belong to the alphabet A) such that any sequence over A is S-adic (see [8]).

In [16], before Cassaigne's constructions, S. Ferenczi used some other techniques to prove a kind of "only if" part of the conjecture for a weaker version of S-adicity. Indeed, he proved that the language of a uniformly recurrent sequence $\mathbf{w}$ with an at most linear complexity is S-adic in the sense that for any factor u of $\mathbf{w}$, there is a non-negative integer n such that u is a factor of $\sigma_0\sigma_1 \cdots \sigma_n(a)$ with $\sigma_0\sigma_1 \cdots \sigma_n \in S^*$. Theorem 1.2 states precisely this result which was originally expressed in terms of symbolic dynamical systems. In this paper, we avoid the language of dynamical systems and try to highlight all the key points of the proof of Theorem 1.2. Then, adapting Ferenczi's methods, we improve this result by proving Theorem 1.1 and give some properties on the S-adic representation that could help stating the condition C. In particular, we show that the constructions used make sense in a more general case and are particularly efficient for sequences with an at most linear complexity.

Theorem 1.1. *Let $\mathbf{w}$ be an aperiodic and uniformly recurrent sequence over an alphabet A. If $\mathbf{w}$ has an at most affine complexity then $\mathbf{w}$ is an S-adic sequence satisfying Properties* 1–5 (*see Section* 6.2) *for a finite set S of non-erasing morphisms such that for all letters a in A, the length of $\sigma_0\sigma_1 \cdots \sigma_n(a)$ tends to infinity with n with $(\sigma_n)_n \in S^{\mathbb{N}}$ (this property will be called the ω-growth Property).*

Theorem 1.2. *(See Ferenczi [16].) Let $\mathbf{w}$ be an aperiodic and uniformly recurrent sequence over an alphabet A with an at most affine complexity. There exist a finite number of morphisms σ_i, $1 \leqslant i \leqslant c$, over an alphabet $D = \{0, \ldots, d-1\}$, an application α from D to A and an infinite sequence $(i_n)_{n\in\mathbb{N}} \in \{1, \ldots, c\}^{\mathbb{N}}$ such that $\inf_{0\leqslant r\leqslant d-1} |\sigma_{i_0}\sigma_{i_1} \cdots \sigma_{i_n}(r)|$ tends to infinity if n tends to infinity and any factor of $\mathbf{w}$ is a factor of $\alpha\sigma_{i_0}\sigma_{i_1} \cdots \sigma_{i_n}(0)$ for some n.*

This paper is organized as follows. Section 2 recalls the definition of S-adicity. In Section 3, we present some results and examples about the conjecture and about the complexity of some particular S-adic sequences. In particular, using a technique similar to the technique in [13] we give an upper bound for the complexity of some S-adic sequences. Section 4 deals with Rauzy graphs. We recall their definition and explain how they evolve. Section 5 presents Ferenczi's methods in a general case and Section 6 gives the proof of Theorem 1.1. We conclude the paper with some remarks in Section 7.

2. *S*-adicity

Let us recall some basic definitions.

An *alphabet* is a finite set A whose elements are called *letters* (or *symbols*). A *word* u over A is a finite sequence of elements of A. The length ℓ of a word $u = u_1 \cdots u_\ell$ is the number of letters of u; it is denoted by $|u|$. The unique word of length 0 is called the *empty word* and is denoted by ε. The set of words of length ℓ over A is denoted by A^ℓ and $A^* = \bigcup_{\ell\in\mathbb{N}} A^\ell$ denotes the set of words over A. The set $A^* \setminus \{\varepsilon\}$ of non-empty words over A is denoted by A^+. The *concatenation* of two words u and v is simply the word uv; u^n is the concatenation of n copies of u. With concatenation, A^* is the free monoid generated by A.

A *sequence* (or *right infinite word*) over A is an element of $A^{\mathbb{N}}$. Recall that with the product topology, the set of sequences $A^{\mathbb{N}}$ is a compact metric space. In the sequel, sequences will be denoted by bold

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