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Noncommutative biorthogonal polynomials

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ABSTRACT

We define and study biorthogonal sequences of polynomials over noncommutative rings, generalizing previous treatments of biorthogonal polynomials over commutative rings and of orthogonal polynomials over noncommutative rings. We extend known recurrence relations for specific cases of biorthogonal polynomials and prove a general version of Favard's theorem.

1. Introduction

The theory of orthogonal polynomials is well established and has many applications. For any sequence $\{S_i\}$ of elements of a commutative ring R, we can define a biadditive function $\langle\cdot,\cdot\rangle : R[x]\times R[x]\to R$ by $\langle ax^i, bx^j\rangle = abS_{i+j}$ for $a,b\in R$ and define a sequence of polynomials $\{p_n\}$ by

$$p_n = \begin{vmatrix} S_n & \cdots & S_{2n-1} & x^n \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & 1 \end{vmatrix}.$$

Then $\langle p_n, p_m\rangle = 0$ if and only if $n \neq m$, i.e. the sequence $\{p_n\}$ is orthogonal. The S_i are called the moments of $\{p_n\}$. For a more detailed introduction see either [3] or [13], Chihara's and Szego's classic texts on the subject. The idea of orthogonal polynomials and this method of generating them has been generalized in two ways to achieve new types of polynomials: noncommutative orthogonal polynomials and biorthogonal polynomials.

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The theory of orthogonal polynomials has been extended to cover rings of noncommutative operators, in particular matrices. The study of orthogonal matrix polynomials started with Krein, for instance in [10] and [11], and developed significantly in the mid 1990s; for example in [4,5,12,6] by Duran, Van Assche and others. In [8], Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon, using the notion of quasideterminants introduced in [9] (see also [7]), extended the theory to general noncommutative rings by setting p_n equal to the quasideterminant of a similar matrix. The paper also shows that the 3-term recurrence relation, which is well-known for commutative orthogonal polynomials, still holds in this case.

Second, orthogonal polynomials have been generalized in several ways to biorthogonal polynomials. See [2] for more details on these generalizations. One such extension is considered in [1] by Bertola, Gekhtman and Szmigielski. A family of biorthogonal polynomials is defined to be two sequences of real polynomials $\{p_n(x)\}$ and $\{q_m(y)\}$ with the property that $\iint p_n(x)q_m(y)K(x,y)\,d\alpha(x)\,d\beta(y) = 0$ when $n \neq m$ for particular K, α and β. In this paper, it is shown that these polynomials can be represented as determinants of matrices whose entries are bimoments and, for a specific $K(x,y)$, a 4-term recurrence relation is obtained.

Here, we define biorthogonal polynomials over a noncommutative ring. We bring together the two different generalizations described above to present a completely algebraic definition of noncommutative biorthogonal polynomials. For our purposes, a biorthogonal family consists of two sequences of polynomials $\{p_n(x)\}$ and $\{q_n(y)\}$, over a division ring R, along with a function $\langle\cdot,\cdot\rangle$: $R[x] \times R[y] \to R$ so that $\langle p_n(x), q_m(y)\rangle = 0$ for all $n \neq m$. Using this definition, we obtain recurrence relations for some types of biorthogonal polynomials and thus generalize the 4-term recurrence relations of [1]. We conclude with a broad extension of Favard's theorem.

2. Set-up and definitions

Let R be a division ring with center C. We will view $R[x]$ as an R-C bimodule of R and $R[y]$ as a C-R bimodule of R. That is, elements of $R[x]$ will be of the form $\sum a_i x^i$ and elements of $R[y]$ will be of the form $\sum y^j b_j$ so that $xc = cx$ and $yc = cy$ for all $c \in C$. Let $\langle\cdot,\cdot\rangle : R[x] \times R[y] \to R$ so that

$$\left\langle \sum a_i x^i, \sum y^j b_j \right\rangle = \sum a_i \langle x^i, y^j \rangle b_j.$$

A system of polynomials $\{p_n\}, \{q_n\}_{n\in\mathbb{N}}$ is *biorthogonal* with respect to $\langle\cdot,\cdot\rangle$ if $\langle p_n(x), q_m(y)\rangle = 0$ for all $n \neq m$.

Let $I_{a,b} = \langle x^a, y^b \rangle$. The set $I = \{I_{a,b}\}_{a,b\in\mathbb{Z}_{\geqslant 0}}$ is called the set of *bimoments* for $\langle\cdot,\cdot\rangle$. The bimoments completely define the function $\langle\cdot,\cdot\rangle$ so we will say that a set of polynomials is biorthogonal with respect to I. In keeping with the notation of [1], we will let I be the matrix of bimoments and write Id for the identity matrix. Note in these cases, and below, all matrices and vectors are infinite, with rows and columns indexed by $\mathbb{Z}_{\geqslant 0}$.

We extend $\langle\cdot,\cdot\rangle$ to $R[x]^n \times R[y]$ and to $R[x] \times R[y]^n$ in the following way:

$$\text{if } B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in R[x]^n \quad \text{and} \quad g \in R[y], \quad \text{then } \langle B, g\rangle = \begin{bmatrix} \langle b_1, g\rangle \\ \vdots \\ \langle b_n, g\rangle \end{bmatrix}.$$

Similarly,

$$\text{if } f \in R[x] \quad \text{and} \quad D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in R[y]^n, \quad \text{then } \langle f, D\rangle = \begin{bmatrix} \langle f, d_1\rangle \\ \vdots \\ \langle f, d_n\rangle \end{bmatrix}.$$

If $C \in Mat_{r\times n}(R)$, $B \in R[x]^n$ and $g \in R[y]$, then $\langle CB, g\rangle = C\langle B, g\rangle$.