Advances in Applied Mathematics 48 (2012) 121–141

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Moments of products of elliptic integrals

James G. Wan

Centre for Computer-Assisted Research Mathematics and its Applications (CARMA), The University of Newcastle, Callaghan, NSW 2308, Australia

ARTICLE INFO

Article history:
Received 28 January 2011
Accepted 30 April 2011
Available online 22 June 2011

MSC:
primary 33C75
secondary 33C20, 42A16, 60G50

Keywords:
Elliptic integrals
Moments
Generalised hypergeometric series

ABSTRACT

We consider the moments of products of complete elliptic integrals of the first and second kinds. In particular, we derive new results using elementary means, aided by computer experimentation and a theorem of W. Zudilin. Diverse related evaluations, and two conjectures, are also given.

1. Motivation and general approach

We study the complete elliptic integral of the first kind, $K(x)$, and the second kind, $E(x)$, defined by:

Definition 1.

$$K(x) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\middle| x^2\right), \qquad E(x) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\middle| x^2\right). \tag{1}$$

As usual, $K'(x) = K(x')$, $E'(x) = E(x')$, where $x' = \sqrt{1-x^2}$. Recall that ${}_pF_q$ denotes the *generalised hypergeometric series*,

$${}_pF_q\left(\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. \tag{2}$$

E-mail address: james.wan@newcastle.edu.au.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2011.04.007

The complete elliptic integrals, apart from their theoretical importance in arbitrary precision numerical computations [6] and the theory of theta functions, are also of significant interest in applied fields such as electrodynamics [18], statistical mechanics, and random walks [9,10]. Indeed, they were first used to provide explicit solutions to the perimeter of an ellipse (among other curves) as well as the (exact) period of an ideal pendulum.

The author was first drawn to the study of integral of products of K and E in [9], in which it is shown that

$$2\int_0^1 K(x)^2\,\mathrm{d}x = \int_0^1 K'(x)^2\,\mathrm{d}x, \tag{3}$$

by relating both sides to a moment of the distance from the origin in a four step uniform random walk on the plane.

A much less recondite proof was only found later: set $x = (1-t)/(1+t)$ on the left-hand side of (3), and apply the quadratic transform (4) below, and the result readily follows.

The four quadratic transforms [6], which we will use over and over again, are

$$K'(x) = \frac{2}{1+x}K\left(\frac{1-x}{1+x}\right), \tag{4}$$

$$K(x) = \frac{1}{1+x}K\left(\frac{2\sqrt{x}}{1+x}\right), \tag{5}$$

$$E'(x) = (1+x)E\left(\frac{1-x}{1+x}\right) - xK'(x), \tag{6}$$

$$E(x) = \frac{1+x}{2}E\left(\frac{2\sqrt{x}}{1+x}\right) + \frac{1-x^2}{2}K(x). \tag{7}$$

In the following sections we will consider definite integrals involving products of K, E, K', E', especially the moments of the products. A goal of this paper is to produce closed forms for these integrals whenever possible. When this is not achieved, closed forms for certain linear combinations of integrals are instead obtained. Thus, we are able to prove a large number of experimentally observed identities in [3].

The somewhat rich and unexpected results lend themselves for easy discovery, thanks to the methods of experimental mathematics: for instance, the *integer relations algorithm* PSLQ [12], the *Inverse Symbolic Calculator* (ISC, now hosted at CARMA [14]), the *Online Encyclopedia of Integer Sequences* (OEIS [17]), the *Maple* package gfun, *Gosper's algorithm* (which finds closed forms for indefinite sums of hypergeometric terms [15]), and Sister Celine's method [15]. Indeed, large scale computer experiments [3] reveal that there is a huge number of identities in the flavour of (3). Once discovered, many results can be routinely established by the following elementary techniques:

(1) Connections with and transforms of hypergeometric and Meijer G-functions [18], as in the case of random walk integrals (Section 3).
(2) Interchange order of summation and integration, which is justified as all terms in the relevant series are positive (Section 4).
(3) Change the variable x to x', usually followed by a quadratic transform (Section 5).
(4) Use a Fourier series (Section 6).
(5) Apply Legendre's relation (Section 7).
(6) Differentiate a product and integrate by parts (Section 8).

Note that Section 2 and most of Section 7 are expository. The propositions in Section 4 are well known, but the arithmetic nature of the moments, Theorem 3 and Lemma 3 in Section 6 do not seem to have featured in previous literature. Section 3 contains new general formulae for the moments of