Advances in Applied Mathematics 48 (2012) 164–180

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

An analogue of the Harer–Zagier formula for unicellular maps on general surfaces

Olivier Bernardi [1]

MIT, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 21 November 2010
Accepted 30 June 2011
Available online 25 August 2011

MSC:
05A15
05A19
05C30

Keywords:
One-face map
Non-orientable surface
Bijection
Moments of the GOE

ABSTRACT

A unicellular map is the embedding of a connected graph in a surface in such a way that the complement of the graph is simply connected. In a famous article, Harer and Zagier established a formula for the generating function of unicellular maps counted according to the number of vertices and edges. The keystone of their approach is a counting formula for unicellular maps on orientable surfaces with n edges, and with vertices colored using every color in $[q]$ (adjacent vertices are authorized to have the same color). We give an analogue of this formula for general (locally orientable) surfaces.

Our approach is bijective and is inspired by Lass's proof of the Harer–Zagier formula. We first revisit Lass's proof and twist it into a bijection between unicellular maps on orientable surfaces with vertices colored using every color in $[q]$, and maps with vertex set $[q]$ on orientable surfaces *with a marked spanning tree*. The bijection immediately implies Harer–Zagier's formula and a formula by Jackson concerning bipartite unicellular maps. It also shed a new light on constructions by Goulden and Nica, Schaeffer and Vassilieva, and Morales and Vassilieva. We then extend the bijection to general surfaces and obtain a correspondence between unicellular maps on general surfaces with vertices colored using every color in $[q]$, and maps on orientable surfaces with vertex set $[q]$ *with a marked planar submap*. This correspondence gives an analogue of the Harer–Zagier formula for general surfaces. We also show that this formula implies a recursion formula due to Ledoux for the numbers of unicellular maps with given numbers of vertices and edges.

E-mail address: bernardi@math.mit.edu.

[1] Supported by the ANR project A3 and the European project ExploreMaps – ERC StG 208471.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2011.06.005

1. Introduction

A *map* is a cellular embedding of a connected graph on a surface considered up to homeomorphism (see Section 3 for definitions). A *planar map* is a map on the sphere. A map is said *unicellular* if it has a single face. For instance, the planar unicellular maps are the plane trees. A map is said *orientable* if the underlying surface is orientable.

In [7] Harer and Zagier considered the problem of enumerating orientable unicellular maps according to the number of edges and vertices (or, equivalently by Euler formula, according to the number of edges and the genus). The keystone of their approach is an enumerative formula for unicellular maps with *colored* vertices (these colorings are not necessarily proper, that is, two adjacent vertices can have the same color): they proved that the number of rooted unicellular maps on orientable surfaces having n edges and vertices colored using *every* color in $[q] := \{1, \dots, q\}$ is

$$T_n(q) = 2^{q-1}\binom{n}{q-1}(2n-1)!!, \tag{1}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 1$.

From (1), it follows that the numbers $\epsilon_v(n)$ of unicellular maps on orientable surfaces with n edges and v vertices satisfy

$$\sum_{v=1}^{n+1} \epsilon_v(n) N^v = \sum_{q=1}^{n+1} \binom{N}{q} 2^{q-1}\binom{n}{q-1}(2n-1)!!. \tag{2}$$

Since both sides of the equation represent the number of unicellular maps with vertices colored using *some* of the colors in $[N]$ (since maps with n edges have at most $n+1$ vertices, and the index q on the right-hand side corresponds to the number of colors really used). From this equation, Harer and Zagier obtained a recurrence relation for the numbers $\epsilon_v(n)$ of orientable unicellular maps with n edges and v vertices:

$$(n+1)\epsilon_v(n) = (4n-2)\epsilon_{v-1}(n-1) + (n-1)(2n-1)(2n-3)\epsilon_v(n-2).$$

The proof of (1) in [7] used a matrix integral argument (see [9] for an accessible presentation). A combinatorial proof was later given by Lass [10], and subsequently a bijective proof was given by Goulden and Nica [6].

The goal of this paper is to give an analogue of Eq. (1) for unicellular maps on *general* (i.e. locally orientable) surfaces, and to describe the bijections hiding behind it (see Eq. (4) below). In order to do so, we first simplify slightly the proof of Lass for the orientable case, and then show how to extend it in order to deal with the general case.

In Section 3, we revisit Lass's proof of (1) in order to obtain a bijection Φ between orientable unicellular maps colored using every color in $[q]$ and orientable *tree-rooted maps* (maps with a marked spanning tree) with vertex set $[q]$. This bijective twist given to Lass's proof turns out to simplify certain calculations because tree-rooted maps are easily seen to be counted by the right-hand side of (1). As mentioned above, a bijection was already exhibited by Goulden and Nica for unicellular maps on orientable surfaces in [6]. However, this bijection is presented in terms of the permutations (in particular the image of the bijection is not expressed in terms of maps) which makes its definition and analysis more delicate. By contrast, it is immediate to see that the bijection Φ satisfies the following property:

($\star$) for any unicellular map U colored using every color in $[q]$ and for any colors $i, j \in [q]$, the number of edges of U with endpoints of color i, j is equal to the number of edges between the vertices i, j in the tree-rooted map $\Phi(U)$.