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Prime divisors of binary holonomic sequences

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Abstract

We prove that if $(u_n)_{n\geqslant 0}$ is a sequence of rational numbers satisfying a recurrence of the type

$$f_0(n)u_{n+2} + f_1(n)u_{n+1} + f_2(n)u_n = 0,$$

where $f_i(X) \in \mathbb{Q}[X]$ are not all zero for $i = 0, 1, 2$, which is not binary recurrent for all sufficiently large n, then there exists a positive constant c depending on the sequence $(u_n)_{n\geqslant 0}$ such that the product of the numerators and denominators of the nonzero rational numbers u_n for all $n \leqslant N$ has at least $c \log N$ prime factors as $N \to \infty$.

MSC: 11A07; 11B65; 11L40

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1. Introduction

Let $(u_n)_{n\geqslant 0}$ be a sequence of rational numbers satisfying a recurrence of the type

$$f_0(n)u_{n+2} + f_1(n)u_{n+1} + f_2(n)u_n = 0 \quad \text{for } n = 0, 1, \ldots, \tag{1}$$

where $f_i(X) \in \mathbb{Q}[X]$ for $i = 0, 1, 2$ not all zero. Such a sequence is called *binary polynomially recurrent*, or *binary holonomic*. Assume further that $(u_n)_{n\geqslant 0}$ is not binary recurrent from some

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point on. By this, we mean that there do not exist integers a, b and c not all zero and $n_0 > 0$ such that

$$au_{n+2} + bu_{n+1} + cu_n = 0$$

holds identically for all $n > n_0$. Write $u_n = a_n/b_n$, where $a_n, b_n \geqslant 1$ are coprime integers. In this paper, we look at the number of prime factors of the number

$$U(N) = \prod_{\substack{n \leqslant N \\ a_n \neq 0}} a_n b_n.$$

For a nonzero integer m let $\omega(m)$ be the number of distinct prime factors of m. We prove the following result.

Theorem 1. *Under the previous assumptions, there exists a positive constant c depending on the sequence $(u_n)_{n \geqslant 0}$ such that the inequality*

$$\omega\big(U(N)\big) > c \log N \tag{2}$$

holds for all $N > 1$.

Recall that for $n \geqslant 1$, the nth Motzkin number m_n counts the number of lattice paths in the Cartesian plane starting at $(0, 0)$, ending at $(n, 0)$, and which use line steps equal to either $(1, 0)$ (level step), or to $(1, 1)$ (up step), or to $(1, -1)$ (down step), and which never pass below the x-axis. It is well known that $m_1 = 1$, $m_2 = 2$ and that $(m_n)_{n \geqslant 1}$ satisfies recurrence (1) for all $n \geqslant 1$ with $f_0(X) = X + 2$, $f_1(X) = 2X + 1$ and $f_2(X) = 3(X - 1)$. In [3], it was shown that if we write

$$M_N = \prod_{n=1}^{N} m_n,$$

then

$$\omega(M_N) \geqslant 10^{-4} \log N \quad (N \geqslant 1)$$

and it was remarked that the same method yields a similar result when the sequence $(m_n)_{n \geqslant 1}$ of Motzkin numbers is replaced by the sequence $(s_n)_{n \geqslant 1}$ of Schröder numbers. Hence, our Theorem 1 above shows that the above inequality on the number of distinct prime factors of such terms up to N holds for all binary holonomic sequences which are not binary recurrent from some point on. It is clear that we need to neglect binary recurrences since geometrical progressions are of this type and inequality (2) fails for them.

In the same context, but by a different method, we recall that Shparlinski [4] showed that if B_n is the nth Bell number and

$$B(N) = \prod_{n \leqslant N} B_n,$$