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Including homoclinic connections and T-point heteroclinic cycles in the same global problem for a reversible family of piecewise linear systems

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ABSTRACT

Apart from being a complex task even in piecewise linear systems, the proof of the existence of homoclinic connections and T–point heteroclinic cycles must be usually carried out in separate ways because they are obviously different dynamical objects. Despite this, some features of the system may narrow the disparities between such global bifurcations and help us to look for alternative methods to analyze them.

In this work, taking advantage of the reversibility and some geometrical features of a piecewise linear version of the Michelson system, we construct, by adding a suitable parameter, a global problem that includes homoclinic connections and T-point heteroclinic cycles as particular cases. Moreover, this problem leads to a common result for the existence and local uniqueness of these global bifurcations, whose proof is also given.

1. Introduction

The importance of homoclinic orbits and T-point heteroclinic cycles in dynamical systems lies in the fact that their existence usually forces a rich and complex behavior. Unfortunately, proving that a dynamical system has such global connections is generally a difficult task. Nevertheless, in this work we do not just prove the existence of these objects in a piecewise linear system, but we even deduce it as a consequence of a general common result.

In a parameterized three-dimensional dynamical system, a homoclinic connection to a saddle equilibrium point appears when the one-dimensional invariant manifold lies in the two-dimensional one (see [9,11,18]). This is generically a phenomenon of codimension-one. On the other hand, a T-point heteroclinic cycle appears when the one-dimensional invariant manifolds of two saddle equilibria are connected and the two-dimensional manifolds intersect (see [1,8,10]). Generically, such a global connection, which is also called Bykov cycle by some other authors [7,19,20], corresponds to a codimension-two set of the parameter space. Obviously, homoclinic connections and T-point heteroclinic cycles are clearly different objects.

Under certain reversibility or symmetry conditions, some of the differences between homoclinic connections and T-point heteroclinic cycles can be reduced. For instance, the reversibility of the Michelson system [17] implies that the codimension of these two global objects coincides [14]. This fact also occurs in the continuous piecewise linear version of the Michelson

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system

$$\begin{cases}\dot{x}=y,\\ \dot{y}=z,\\ \dot{z}=1-y-\lambda(1+\lambda^2)|x|,\end{cases} \tag{1}$$

where the parameter λ is strictly positive. More details can be found in [2–6].

System (1) is time-reversible with respect to the involution $\mathcal{R}(x,y,z)=(-x,y,-z)$, whose set of fixed points corresponds to the y–axis, which is called reversibility axis. The system is given by two linear systems separated by the plane $\{x=0\}$, which is called the separation plane, and it can be written in matrix form as

$$\dot{\mathbf{x}}=\begin{cases}A^-\mathbf{x}+\mathbf{e}_3 & \text{if } x\le 0,\\ A^+\mathbf{x}+\mathbf{e}_3 & \text{if } x\ge 0,\end{cases}$$

with $\mathbf{x}=(x,y,z)^T$, $\mathbf{e}_3=(0,0,1)^T$ and

$$A^\pm=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \mp\lambda(1+\lambda^2) & -1 & 0\end{pmatrix}.$$

In the half–space $\{x<0\}$, system (1) has exactly one equilibrium point $\mathbf{p}^-=(-1/(\lambda+\lambda^3),0,0)^T$, which is a saddle-focus point given that the eigenvalues of the matrix A^- are λ, $\alpha\pm i\beta$, with

$$\alpha=-\frac{\lambda}{2},\quad \beta=\frac{\sqrt{4+3\lambda^2}}{2}. \tag{2}$$

By the reversibility with respect to $\mathcal{R}$, there exists exactly one saddle-focus equilibrium $\mathbf{p}^+=-\mathbf{p}^-$ in the half–space $\{x>0\}$, whose eigenvalues are given by $-\lambda$ and $-\alpha\pm i\beta$.

In piecewise linear systems, global connections can be classified in a natural way attending to the number of points of intersection with the separation boundary. Global connections with the least number of intersections will be called direct. In particular, a direct homoclinic connection of system (1) intersects the separation plane $\{x=0\}$ at exactly two points and a direct reversible T-point heteroclinic cycle at four (three of them correspond to the one-dimensional invariant manifolds and the other one to the two-dimensional invariant manifolds).

Since a direct homoclinic connection and a direct reversible T-point heteroclinic cycle are different objects, in [2] and [3], different proofs were given for the existence and the local uniqueness of these direct global connections of system (1). In any case, we must remark that the techniques and some details were similar. All of this, together with later readings of both works, leads us to wonder if it would be possible to include both results in a common theorem. The answer to this question is at the core of this work.

The paper is organized as follows. Section 2 is devoted to establish the main Theorem of this paper and its application to prove the existence and local uniqueness of direct homoclinic connections and direct reversible T-point heteroclinic cycles of system (1). In order to do this, we remind the conditions, previously obtained in [2] and [3], that these direct global connections must satisfy. From them, by introducing an additional parameter, we deduce a common set of conditions. Section 3 is devoted to prove the main result and, for the sake of simplicity, some technical results are relegated to an appendix. Finally, some lines about future works have been added.

2. Common set of conditions for the global connections. Establishment of the main theorem

As it has been mentioned above, the existence of a direct homoclinic connection and the existence of a direct reversible T-point heteroclinic cycle in system (1) were separately proven in [2,3]. An essential step in both proofs was to obtain a suitable set of conditions these global connections must satisfy. In this section, both sets of conditions are described separately, in terms of the Poincaré half–maps, for the purpose of showing their similarity and establishing a common set of conditions.

For piecewise linear systems with two linearity zones, a Poincaré half–map can be defined in each zone. Roughly speaking, in system (1), the Poincaré half–map defined at the half–space $\{x<0\}$ maps points from the set $\Sigma^-=\{x=0,y<0\}$ to points in the set $\Sigma^+=\{x=0,y>0\}$ by means of the flow of the system. Analogously, the other Poincaré half–map assigns points from Σ^+ to points in Σ^-.

A more rigorous definition of the Poincaré half–maps in other piecewise linear systems can be found in [12,15,16].

A direct homoclinic orbit to the equilibrium point $\mathbf{p}^-$ of system (1) is one that intersects the separation plane $\{x=0\}$ at exactly two points. Since this system is linear in each zone, the one–dimensional invariant manifold of $\mathbf{p}^-$ is locally a straight line in the half–space $\{x<0\}$. This straight line is generated by the eigenvector associated with the real eigenvalue λ, passes through the equilibrium point $\mathbf{p}^-$ and intersects the separation plane at $\mathbf{m}^-=\left(0,1/(\lambda^2+1),\lambda/(\lambda^2+1)\right)^T$, which is shown in Fig. 1. The solution of system (1) with parameter λ and initial condition $\mathbf{x}(0,\lambda)=\mathbf{m}^-$ is denoted by $\mathbf{x}(t,\lambda)=(x(t,\lambda),y(t,\lambda),z(t,\lambda))^T$.