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Discrete spline methods for solving two point fractional Bagley–Torvik equation

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ABSTRACT

A new discrete spline method is developed to solve the two point fractional Bagley–Torvik equation.The method is based on discrete spline function and a nonstandard Grünwald–Letnikov difference (NSGD) and the weighted and shifted Grünwald–Letnikov difference (WSGD) operators to approximate the fractional derivative. Bounds for Grünwald–Letnikov weights are considered. Convergence analysis is discussed and a class of second order and third order methods are obtained. Illustrative examples are presented to validate the practical usefulness of the methods.

1. Introduction

We consider the following Bagley–Torvik equation:

$$u^{(2)}(x) + \theta D^{\alpha} u(x) + \sigma u(x) = f(x), m-1 \le \alpha < m, \quad x \in [0, 1], \tag{1}$$

subject to boundary conditions:

$$u(0) = u(1) = 0, \tag{2}$$

where the function $f(x)$ is continuous on the interval [0, 1] and the operator D^{α} represents the Caputo fractional derivative.

In general, this type of fractional boundary value problem arises in many applications in science and engineering such as viscoelasticity, fluid mechanics, vibration, digital control theory, biology, and bioengineering.

There are a lot of papers in the literature dealing with initial fractional Bagley–Torvik equation, We may refer to [2–4,9,10,12,13,17,19,29]. Fitt et al. [7] developed a fractional mathematical model for micro-electro-mechanical system (MEMS) instrument to measure the viscosity of fluids during the process of oil well exploration. Mashayekhi and Razzaghi [34], introduced a numerical method for solving the initial fractional Bagley–Torvik equation, this method is based upon block-pulse functions and Bernoulli polynomials. In [24] Yüzbasi presented a numerical technique for solving the Bagley–Torvik equation via a generalized form of the Bessel functions of the first kind. Ray [35], solved initial fractional Bagley–Torvik equation by operational matrix of Haar wavelet method. Gülsu et. al. [33], presented a numerical solution method to approximate the solution of the initial Bagley–Torvik equation of fractional order in terms of the generalized Taylor series. Gómez et al. [8] discussed the fractional mass-spring-damper system of the Caputo type and introduced a parameter that

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characterized the existence of fractional components in the system. In [25], Wang and Wang investigated the general solution of the initial value Bagley–Torvik equation by changing it into a sequential fractional differential equation with constant coefficients. Then the general solution is expressed as the linear combination of fundamental solutions that are in terms of α-exponential functions. Scherer et. al. [20], devoted a numerical treatment of initial fractional differential equations using Grünwald–Letnikov definition of fractional derivatives and discussed the asymptotic stability and the absolute stability of the proposed methods. Cérmák and Kisela [32], discussed in details stability and asymptotic properties of the initial fractional Bagley–Torvik equation for the exact as well as numerical solutions obtained by using Grünwald–Letnikov discretization of the problem.

On the other hand , the literature contains little on solving two point boundary value Bagley–Torvik equations. Stanek [21] discussed existence and uniqueness results for two point boundary value Bagley–Torvik equation. Also, in Stynes and Gracia [22] another class of two-point boundary value problem whose highest order term is a Caputo fractional derivative of order $\delta \in (1, 2)$ is discretized using a finite difference method. Dimitrov [5] used a combination of Grünwald–Letnikov and shifted Grünwald–Letnikov to propose a second order and third order approximations for the Caputo fractional derivative and then used these approximations to solve diffusion problems. Also, Tian et. al. [23] discussed a class of second order difference approximations based on the weighted and shifted Grünwald–Letnikov difference operators for Riemann–Liouville fractional derivative with their applications to solve space fractional diffusion equations. Finally, Zhao-peng et. al. [31] derived a fourth order difference approximations for Riemann–Liouville fractional derivative and applied this method to solve the space fractional diffusion equations. For more details about the approximations of the space fractional derivatives, we may refer to [5,23,31] and the reference therein. In Zahra and Elkholy [26,28], the authors introduced two techniques for solving the Bagley–Torvik equation with initial and boundary conditions. The first approach is based on transforming the fractional derivative into a system of ordinary differential equations and then using cubic polynomial spline with shooting method to find the approximate solution of the problem. The second one depends on approximating the fractional term using the Grünwald–Letnikov definition of the fractional derivative. Also, in Zahra and Elkholy [27] quadratic polynomial spline function was considered to find an approximate solution for a class of boundary value problems of fractional order.

As we know a continuous spline function needs derivatives at some points. But, in the real world, it may be difficult to compute the derivatives and in that case usual spline will not be suitable. So the main object of this paper is to construct new discrete spline Numerov methods based on central differences to obtain an approximation for the solution of fractional boundary value problem (1 and 2). In Lyche [11], most of the theory of polynomial splines deals with the case where the pieces are tied together by continuity of certain derivatives at the knots but in discrete spline the ties will involve differences instead of derivatives. For more details about discrete spline, see [1,6,11,30] and the references therein. The paper is organized as follows: In Section 2, we briefly review the main definitions of fractional derivatives and the discrete cubic spline function. Derivation of the method and convergence analysis are discussed in Section 3. Numerical results are presented to illustrate the applicability and accuracy in Section 4. Finally, in Section 5 we conclude the results of the proposed methods.

2. Basic definitions

In this section, we illustrate the basic definitions and properties in the theory of fractional calculus; moreover we introduce the principle of discrete cubic spline [1,11].

2.1. Fractional calculus

There are many approaches for fractional derivatives, we focus our attention to the following definitions, see [4,9,12,16,18].

Definition 1. The Riemann–Liouville (R–L) fractional derivative of $y(t)$ is:

$$^{R}D^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_a^t (t-\tau)^{m-\alpha-1} y(\tau)\, d\tau, & m-1 \le \alpha < m \\ y^{(m)}(t), & \alpha = m, \end{cases} \tag{3}$$

where $\Gamma(.)$ is the gamma function, $m = \lceil \alpha \rceil$ is the smallest integer such that $m>\alpha$ and d^m/dx^m denotes the standard derivatives of integer order.

Definition 2. The Caputo fractional derivative of $y(t)$ is:

$$D^{\alpha} y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{y^{(m)}(\tau)}{(t-\tau)^{-m+\alpha+1}}\, d\tau, & m-1 \le \alpha < m \\ y^{(m)}(t), & \alpha = m. \end{cases} \tag{4}$$

Relation between Riemann–Liouville and Caputo operators

The Caputo operator D^{α} has advantages for fractional differential equations (FDEs) with initial conditions. It is possible to couple fractional differential equations in the sense of Caputo with classical initial conditions while FDEs in the sense of Riemann–Liouville are accompanied with initial conditions expressed in fractional derivatives. Since there is no clear