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On spectral radius and energy of extended adjacency matrix of graphs

Kinkar Ch. Das[a,*], Ivan Gutman[b,c], Boris Furtula[b]

[a] Department of Mathematics, Sungkyunkwan University, Suwon, Republic of Korea
[b] Faculty of Science, University of Kragujevac, Kragujevac, Serbia
[c] State University of Novi Pazar, Novi Pazar, Serbia

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ABSTRACT

Let G be a graph of order n. For $i = 1, 2, \dots, n$, let d_i be the degree of the vertex v_i of G. The extended adjacency matrix $\mathbf{A}_{ex}$ of G is defined so that its (i, j)-entry is equal to $\frac{1}{2}(\frac{d_i}{d_j} + \frac{d_j}{d_i})$ if the vertices v_i and v_j are adjacent, and 0 otherwise,Yang et al. (1994). The spectral radius η_1 and the energy $\mathcal{E}_{ex}$ of the $\mathbf{A}_{ex}$-matrix are examined. Lower and upper bounds on η_1 and $\mathcal{E}_{ex}$ are obtained, and the respective extremal graphs characterized.

1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$, and let $|E(G)| = m$. If the vertices v_i and v_j are adjacent, we write $v_iv_j \in E(G)$. For $i = 1, 2, \dots, n$, let d_i be the degree of the vertex v_i. The maximum and minimum degrees of the graph G are denoted by Δ and δ, respectively.

The adjacency matrix $\mathbf{A} = \mathbf{A}(G)$ of the graph G is defined so that its (i, j)-entry is equal to 1 if $v_iv_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$ denote the eigenvalues of $\mathbf{A}(G)$. The greatest eigenvalue λ_1 is usually referred to as the *spectral radius* of the graph G.

The *energy* of the graph G is defined as

$$\mathcal{E} = \mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For details of the mathematical theory of this, nowadays very popular, graph–spectral invariant see the monograph [23], the recent papers [4–7,12,16,21,24,25], and the references cited therein.

The extended adjacency matrix of the graph G, denoted by $\mathbf{A}_{ex} = \mathbf{A}_{ex}(G)$, was put forward by Yang et al. [31] and is defined so that its (i, j)-entry is equal to $\frac{1}{2}(\frac{d_i}{d_j} + \frac{d_j}{d_i})$ if $v_iv_j \in E(G)$ and 0 otherwise.

It is immediately seen that in the case of regular graphs the extended adjacency matrix and the ordinary adjacency matrix coincide.

Since $\mathbf{A}_{ex}$ is a symmetric matrix of order n, all its eigenvalues are real. These are denoted by $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$. Since $\mathbf{A}_{ex}$ is an irreducible non-negative $n \times n$ matrix, its greatest eigenvalue may be viewed as the *extended spectral radius* of the graph G, a quantity first studied by Yang et al. [31].

* Corresponding author. Fax: +82 31 290 7033.
E-mail addresses: kinkardas2003@googlemail.com, kinkar@lycos.com (K.Ch. Das), gutman@kg.ac.rs (I. Gutman), furtula@kg.ac.rs (B. Furtula).

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In the paper [31] also the sum of the absolute values of the eigenvalues of the $\mathbf{A}_{ex}$-matrix were considered. The authors of [31] seem to have not been aware that this was just the graph energy pertaining to the matrix $\mathbf{A}_{ex}$ [27,28]. Anyway, what was introduced in [31] was the *extended graph energy*, defined as

$$\mathcal{E}_{ex} = \mathcal{E}_{ex}(G) = \sum_{i=1}^{n} |\eta_i|. \tag{1}$$

It is worth noting that the extended graph energy was conceived more than ten years before the Laplacian [15,17,23], distance [15,22,23], matching [2,30] and Randić [8,9] energies were put forward. Thus, $\mathcal{E}_{ex}$ happens to be the first and earliest modification of the ordinary (on the adjacency matrix based) graph energy $\mathcal{E}$ [13].

In the later part of this paper we shall need three degree-based graph invariants, namely F, M_2, and M_2^*. The forgotten topological index F is defined as

$$F = F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{v_iv_j \in E(G)} \left(d_i^2 + d_j^2\right). \tag{2}$$

For its basic properties see [11] and the references cited therein. The second Zagreb index M_2 is

$$M_2 = M_2(G) = \sum_{v_iv_j \in E(G)} d_i\, d_j. \tag{3}$$

For its basic properties see [10,14] and the references cited therein. The modified second Zagreb index M_2^* is [26]

$$M_2^* = M_2^*(G) = \sum_{v_iv_j \in E(G)} \frac{1}{d_i\, d_j}. \tag{4}$$

For lower and upper bounds on M_2^*, see [18].

As usual, by $K_{p,\,q}$ $(p+q=n)$, K_n, and $K_{1,n-1}$ we denote, respectively, the complete bipartite graph, the complete graph, and the star on n vertices. For other undefined notations and terminology from graph theory, the readers are referred to [1].

The rest of the paper is structured as follows. In Section 2, we state some previously known results, needed for the subsequent considerations. In Section 3, we give some lower and upper bounds on the extended spectral radius and characterize the extremal graphs. In Section 4, we obtain some lower and upper bounds on the extended graph energy and characterize the extremal graphs.

2. Lemmas

We state here some previously known results that are needed in the next two sections.

Lemma 1 (Rayleigh–Ritz) [32]**.** *If* $\mathbf{C}$ *is a symmetric* $n \times n$ *matrix with eigenvalues* $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_n$, *then for any* $\mathbf{x} \in R^n$, *such that* $\mathbf{x} \neq \mathbf{0}$,

$$\mathbf{x}^T\mathbf{C}\,\mathbf{x} \le \rho_1\, \mathbf{x}^T\mathbf{x}.$$

Equality holds if and only if $\mathbf{x}$ *is an eigenvector of* $\mathbf{C}$ *corresponding to the largest eigenvalue* ρ_1.

Lemma 2 [20]**.** *Let* $\mathbf{C} = (c_{ij})$ *and* $\mathbf{D} = (d_{ij})$ *be symmetric, non-negative matrices of order n. If* $\mathbf{C} \ge \mathbf{D}$, *i.e.,* $c_{ij} \ge d_{ij}$ *for all i, j, then* $\rho_1(\mathbf{C}) \ge \rho_1(\mathbf{D})$, *where* ρ_1 *is the largest eigenvalue.*

Lemma 3 [29]**.** *Let* $\mathbf{C}$ *be a symmetric matrix of order n, and let* $\mathbf{C}_k$ *be its leading* $k \times k$ *submatrix. Then, for* $i = 1, 2, \ldots, k$,

$$\rho_{n-i+1}(\mathbf{C}) \le \rho_{k-i+1}(\mathbf{C}_k) \le \rho_{k-i+1}(\mathbf{C})\,,$$

where $\rho_i(\mathbf{C})$ *is the ith largest eigenvalue of* $\mathbf{C}$.

Lemma 4 [19]**.** *Let G be a connected graph of order n with m edges. Then*

$$\lambda_1(G) \le \sqrt{2m - n + 1}$$

with equality holding if and only if $G \cong K_{1,n-1}$ *or* $G \cong K_n$.

The nullity $n_0(G)$ of a graph G is the multiplicity of the eigenvalue zero in its adjacency spectrum.

Lemma 5 [3]**.** *Let G be a graph of order* $n \ge 2$. *Then* $n_0(G) = n - 2$ *if and only if* $G \cong K_{p,q} \cup (n-p-q)\,K_1$, *where* $p + q \le n$.

Concluding this section we determine two elementary properties of the eigenvalues of the extended adjacency matrix.

Lemma 6. *Let G be a connected graph of order n. Then* $\eta_1 > \eta_2$.

Proof. We prove this result by contradiction. Assume that $\eta_1 = \eta_2$. Since G is connected, by the Perron–Frobenius theorem, the eigenvector $\mathbf{x}$ corresponding to η_1 has all components positive. Let $\mathbf{y}$ be the eigenvector corresponding to η_2. Since