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Maps preserving the diamond partial order

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ABSTRACT

The present paper is devoted to the study of the diamond partial order in C^*-algebras. We characterize linear maps preserving this partial order.

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1. Introduction

Let A be a (complex) Banach algebra. An element a in A is (*von Neumann*) *regular* if it has a generalized inverse, that is, if there exists b in A such that $a = aba$ (b is an *inner inverse* of a) and $b = bab$ (b is an *outer inverse* of a). The generalized inverse of a regular element a might not be unique. Observe also that the first equality $a = aba$ is a necessary and sufficient condition for a to be regular, and that, if a has generalized inverse b, then $p = ab$ and $q = ba$ are idempotents in A with $aA = pA$ and $Aa = Aq$.

We denote by $A^\bullet$ the set of idempotent elements in A and by $A^\wedge$ the set of all regular elements of A.

The unique generalized inverse of a that commutes with a is called the *group inverse* of a, whenever it exists. In this case a is said to be *group invertible* and we write $a^\sharp$ to refer to its group inverse. The set of all group invertible elements of A is denoted by $A^\sharp$.

For an element a in A, the left and right multiplication operators are defined by L_a: $x \mapsto ax$ and R_a: $x \mapsto xa$, respectively. If a is regular, then so are L_a and R_a, and thus their ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed.

Regular elements in unital C*-algebras have been studied by Harte and Mbekhta in [16], where they prove that an element a in a C*-algebra A is regular if and only if aA is closed.

Given a and b in A, b is said to be a Moore–Penrose inverse of a if b is a generalized inverse of a and the associated idempotents ab and ba are selfadjoint (that is, projections). It is known that every regular element a in A has a unique Moore–Penrose inverse that will be denoted by $a^\dagger$ [16, Theorem 6]. Therefore, the Moore–Penrose inverse of a regular element $a \in A$ is the unique element fulfilling the following equations:

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$$aa^{\dagger}a = a, \quad a^{\dagger}aa^{\dagger} = a^{\dagger}, \quad (aa^{\dagger})^* = aa^{\dagger}, \quad (a^{\dagger}a)^* = a^{\dagger}a.$$

In what follows, let us denote by Proj(A) the set of projections of A.

Generalized inverses are used in the study of partial orders on matrices, operator algebras and abstract rings. Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. On $M_n(\mathbb{C})$ there are many classical partial orders (see [2,13,17,18,23–25]). We recall some of these next. The *star partial order* on $M_n(\mathbb{C})$ was introduced by Drazin in [13], as follows:

$$A \leq_* B \text{ if } A^*A = A^*B \text{ and } AA^* = BA^*,$$

where, as usual, A^* denotes the conjugate transpose of A. He showed that $A \leq {}_*B$ if and only if $A^{\dagger}A = A^{\dagger}B$ and $AA^{\dagger} = BA^{\dagger}$. Baksalary and Mitra introduced in [3] the *left-star* and *right-star* partial order on $M_n(\mathbb{C})$, as

$$A* \leq B \quad \text{if} \quad A^*A = A^*B \text{ and } \operatorname{Im}A \subseteq \operatorname{Im}B,$$

and

$$A \leq *B \text{ if } AA^* = BA^* \text{ and } \operatorname{Im}A^* \subseteq \operatorname{Im}B^*,$$

respectively. It follows that, $A \leq {}_*B$ if and only if $A^* \leq B$ and $A \leq {}^*B$.

In [17] Hartwig introduced the *rank substractivity order*, usually known as the *minus partial order* on $M_n(\mathbb{C})$:

$$A \leq^- B \qquad \text{if} \qquad \operatorname{rank}(B - A) = \operatorname{rank}(B) - \operatorname{rank}(A).$$

It is proved that

$$A \leq^- B \qquad \text{if and only if} \qquad A^-A = A^-B \text{ and } AA^- = BA^-,$$

where A^- denotes an inner inverse of A. Later, Mitra used in [23] the group inverse of a matrix to define the *sharp order* on group invertible matrices:

$$A \leq_{\sharp} B \qquad \text{if} \qquad A^{\sharp}A = A^{\sharp}B \text{ and } AA^{\sharp} = BA^{\sharp}.$$

The *diamond partial order* on $M_n(\mathbb{C})$ was defined in [2] by Baksalary and Hauke:

$$A \leq_{\diamond} B \quad \text{if} \quad AA^*A = AB^*A, \ \operatorname{Im}A \subseteq \operatorname{Im}B \text{ and } \operatorname{Im}A^* \subseteq \operatorname{Im}B^*.$$

Their motivation was the formulation of the matrix version of Cochrain statistical theorem. They proved that this is a partial order on $M_n(\mathbb{C})$, related with the minus partial order by

$$A \leq_{\diamond} B \qquad \text{if and only if} \qquad A^{\dagger} \leq^- B^{\dagger}.$$

Determining whether or not two matrices are related with respect to the diamond partial order can be done by using this equivalence and the fact that there are computational methods to calculate the Moore–Penrose inverse of a matrix. Notice that once we know the Moore–Penrose inverse it suffices to compare the ranges.

Let H be an infinite-dimensional complex Hilbert space, and $B(H)$ the C*-algebra of all bounded linear operators on H. Šemrl extended in [28] the minus partial order from $M_n(\mathbb{C})$ to $B(H)$, avoiding inner inverses: for $A, B \in B(H)$, $A \preceq B$ if and only if there exist idempotent operators $P, Q \in B(H)$ such that

$$R(P) = \overline{R(A)}, \quad N(A) = N(Q), \quad PA = PB, \quad AQ = BQ.$$

Recently Djordjević et al. [11] generalized Šemrl's definition to the scope of Rickart rings and generalized some of the known results to this new setting.

Nowadays, one of the most active and fertile research area in Linear Algebra, Operator Theory and Functional Analysis, are the "linear preserver problems": characterize linear maps between algebras that leave certain functions, subsets, relations, or properties invariant. See for instance [15,22] and the references therein.

In [28], Šemrl studied bijective maps preserving the minus partial order. For an infinite-dimensional complex Hilbert space H, a mapping ϕ: $B(H) \to B(H)$ preserves the minus partial order if $A \preceq B$ implies $\phi(A) \preceq \phi(B)$. The map ϕ: $B(H) \to B(H)$ preserves the minus order in both directions whenever $A \preceq B$ if and only if $\phi(A) \preceq \phi(B)$. Šemrl proved that a bijective map ϕ: $B(H) \to B(H)$ preserving the minus partial order in both directions is either of the form $\phi(A) = TAS$ or $\phi(A) = TA^*S$, for some invertible operators T and S (both linear in the first case and both conjugate linear in the second one).

In [1] Alieva and Guterman studied linear maps preserving the star, left-star, right-star and diamond partial orders between matrix algebras. A linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the star partial order if $A \leq {}_*B$ implies $\phi(A) \leq {}_*\phi(B)$. Linear maps preserving left-star, right-star and diamond partial order are defined in a similar way.

Theorem 1.1. *([1, Proposition 6.2, Theorem 6.3]) A linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserving the diamond partial order is either identically zero or bijective. If $T \neq 0$, then there exist $\alpha \in \mathbb{C} \setminus \{0\}$ and unitary matrices $M, N \in M_n(\mathbb{C})$ such that T is either of the form*

$$T(X) = \alpha MXN, \quad \text{for all } X \in M_n(\mathbb{C}),$$

or of the form

$$T(X) = \alpha MX^{tr}N, \quad \text{for all } X \in M_n(\mathbb{C}).$$