Applied Mathematics and Computation 296 (2017) 182–197

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A numerical study of the performance of alternative weighted ENO methods based on various numerical fluxes for conservation law☆

Hongxia Liu

College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, Shanxi, P.R. China

ARTICLE INFO

Keywords:
Finite difference WENO scheme
Numerical flux
Riemann solver
High-order accurate
Hyperbolic conservation law

ABSTRACT

In this paper, we systematically investigate the performance of the weighted essential non-oscillatory (WENO) methods based on various numerical fluxes for the nonlinear hyperbolic conservation law. Our objective is enhancing the performance for the conservation laws through picking out the suitable numerical fluxes. We focus our attention entirely on the comparison of eight numerical fluxes with the fifth-order accurate finite difference WENO methods and third-order accurate TVD Runge–Kutta time discretization for hyperbolic conservation laws. In addition, we give their implementation based on a new form framework of flux which we used in [9] and was proposed by Shu and Osher in [16]. The detailed numerical study is mainly implemented for the one dimensional system case, including the discussion of the CPU cost, accuracy, non-oscillatory property, and resolution of discontinuities. Numerical tests are also performed for two dimensional systems.

1. Introduction

We know that one of the important components of the WENO schemes [6,8,14] for the conservation law is the numerical flux whose performance sometimes can directly affect the numerical result (of pros and cons). In this paper, we study an alternative flux formulation based on the finite difference WENO methods to solve the nonlinear hyperbolic conservation law

$$\begin{cases} u_t + \nabla \cdot f(u) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

The aim is to investigate the performance of different numerical fluxes based on the WENO methods, and to obtain better performance for the hyperbolic conservation laws by choosing suitable numerical fluxes. Moreover, in most of the WENO methods in the literature, the Lax–Friedrichs (LF) numerical flux was used due to its simplicity and easy implementation. Although LF flux is one of the simplest and most widely used for the high order finite difference/volume WENO methods, the numerical viscosity of the LF flux is also the largest among many other monotone fluxes. Thus we cannot get a satisfactory numerical results, since the numerical viscosity may smooth many useful messages with the time development. There are

☆ The research was partially supported by the National Science Foundation of China (No. 11601364) and Special/Youth Foundation of Taiyuan University of Technology (No. 2015QN060).

E-mail address: hx_ryu@163.com

http://dx.doi.org/10.1016/j.amc.2016.10.023

many other numerical fluxes based on either exact or approximate Riemann solvers in the literature [3–5,10–12,19,21], which could also be used in the context of the WENO methods.

We now give a brief survey of WENO schemes. WENO schemes are high order schemes for approximating hyperbolic conservation laws and other convection dominated partial differential equations. They can produce sharp, non-oscillatory discontinuity transitions and high order accurate resolutions for the smooth part of the solution. The first WENO scheme was introduced in 1994 by Liu et al. in their pioneering paper [8], in which a third order accurate finite volume WENO scheme in one space dimension was constructed. In [6], a general framework is provided to design arbitrary high order accurate finite difference WENO schemes, which are more efficient for multidimensional calculations. Very high order finite difference WENO schemes (seventh to eleventh order) are documented in [1].

However, for a finite difference WENO scheme using the flux splitting method to compute its numerical fluxes generally [14], it is difficult to implement the procedure for most two point monotone numerical fluxes because we do have not splitting each of them to satisfy the upwinding performance for the stability of the scheme. In [7], Jiang et al. designed a finite difference WENO scheme with Lax–Wendroff time discretization, avoiding the above mentioned difficulty.

In this work, we would like to explore the alternative formulation for constructing numerical fluxes in high order conservative finite difference schemes which involves interpolations directly on the point values of the solution u_i rather than on the flux values. This alternative formulation, even though less clean and more computationally expensive, overcomes the above disadvantages, while enabling the use of monotone fluxes and interpolation directly on the points value. We take a numerical flux as being composed of two terms, a low and a high order flux. This approach has been proposed in [16] and used recently in [7]. As a low order flux we choose same typically used monotone numerical flux and add high order flux by Taylor expansion.

As summarized in [7], one of the advantages of this form flux is that arbitrary monotone fluxes can be used in this framework, while the traditional practice of constructing flux functions can be applied only to smooth flux splitting. Therefore, we can obtain an alternative formulations of conservation finite difference WENO schemes, where instead of a low order numerical flux $\hat{f}^L_{i+\frac{1}{2}}$ in (2.5) we use eight different numerical fluxes.

The fluxes that are consider in this work, are the Godunov flux [4], the Engquist–Osher (EO) flux [3] (for the scalar case, its extension to systems referred to as the Osher–Solomon flux), the Harten, Lax and van Leer (HLL) flux [5], and a modification of the HLL flux, called HLLC flux. These fluxes are based on the approximation Riemann solver, are two point, first-order monotone fluxes; the others we study are essentially two point fluxes, they satisfy the essential two point property: $\hat{f}(u^l, u, u, u^r) = f(u)$ for any u^l and u^r, which can be viewed as the correction of a low order flux. A comparison is made between the different schemes based on the WENO scheme to approximate the hyperbolic conservation laws mainly for the one dimensional system case. The approximate Riemann solver of LF, HLL and HLLC are used in two dimensional systems. We review and describe the details of the numerical fluxes under consideration in Section 2, and present extensive numerical experiments and compare their performance of these considered fluxes in Section 3. Concluding remarks are given in Section 4.

2. Review and implementation of the numerical fluxes for the WENO methods

In this section we review the WENO method for spatial discretization of the conservation laws and the numerical fluxes under consideration and nonlinear stable Runge–Kutta methods for time. We start with the description of the WENO method in one dimensional scalar case.

Consider the one dimensional hyperbolic conservation law given by

$$u_t + f(u)_x = 0. \tag{2.1}$$

By taking the uniform mesh $x_{i+\frac{1}{2}} = x_i + \frac{\Delta x}{2}$, $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), i = 1, \ldots, N,$ where $\Delta x = \frac{1}{N}$, denoting the cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, a semidiscrete conservative finite difference scheme for solving (2.1) has the form

$$\frac{du_i(t)}{dt} = -\frac{1}{\Delta x}(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}) \tag{2.2}$$

where $u_i(t)$ is the numerical approximation to the nodal value $u(x_i, t)$. The numerical flux

$$\hat{f}_{i+\frac{1}{2}} = \hat{f}(u_{i-r}, \ldots, u_{i+s}) \tag{2.3}$$

is designed by

$$\frac{1}{\Delta x}(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}) = f(u(x))_x|_{x_i} + O(\Delta x^k) \tag{2.4}$$

for a k-order scheme. In order to obtain higher order schemes ($k \geq 3$), the finite difference WENO schemes customarily introduce an integrand $h(x)$, then use the fundamental theorem of calculus, to obtain a high-order accurate numerical flux $\hat{f}_{i+1/2}$. This approach uses Lemma 2.1 in [14], for details see also [6,15]. However, a disadvantage of using Lemma 2.1 is that we need to choose a suitable numerical flux which can split incorporated upwinding of the scheme for achieving the nonlinear stability.