Applied Mathematics and Computation 296 (2017) 198–214

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Collocation methods for Volterra functional integral equations with non-vanishing delays[☆]

Wanyuan Ming[a,b], Chengming Huang[a,*]

[a] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, PR China
[b] School of Mathematics and Information Sciences, Nanchang Hangkong University, Nanchang 330063, PR China

ARTICLE INFO

Keywords:
Volterra functional integral equations
Non-vanishing delays
Collocation methods
Optimal order of superconvergence
θ-invariant meshes

ABSTRACT

In this paper the existence, uniqueness, regularity properties, and in particular, the local representation of solutions for general Volterra functional integral equations with non-vanishing delays, are investigated. Based on the solution representation, we detailedly analyze the attainable (global and local) convergence order of (iterated) collocation solutions on θ-invariant meshes. It turns out that collocation at the m Gauss (-Legendre) points neither leads to the optimal global convergence order $m+1$, nor yields the local convergence order $2m$ on the whole interval, which is in sharp contrast to the case of the classical Volterra delay integral equations. However, if the collocation is based on the m Radau II points, the local superconvergence order $2m-1$ will exhibit at all mesh points. Finally, some numerical experiments are performed to confirm our theoretical findings.

1. Introduction

Consider the following general Volterra functional integral equation (VFIE)

$$\begin{cases} y(t) = g(t) + b(t)y(\theta(t)) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), & t \in I := (t_0, T], \\ y(t) = \phi(t), & t \in I_\theta := [\theta(t_0), t_0], \end{cases} \tag{1.1}$$

where the Volterra operators $\mathcal{V}$ and $\mathcal{V}_\theta$ are given by

$$(\mathcal{V}y)(t) := \int_{t_0}^{t} K_1(t,s)y(s)ds, \quad t \in I, \tag{1.2}$$

and

$$(\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} K_2(t,s)y(s)ds, \quad t \in I, \tag{1.3}$$

respectively. The kernel functions $K_1(t, s)$ and $K_2(t, s)$ in (1.2) and (1.3) are supposed to be continuous on their respective domains $D := \{(t, s): t_0 \le s \le t \le T, t \in I\}$ and $D_\theta := \{(t, s): \theta(t_0) \le s \le \theta(t), t \in I\}$.

☆ Supported by the National Natural Science Foundation of China (Grant Nos. 11371157 and 11401294).
* Corresponding author.
E-mail addresses: mingwanyuan@sina.com (W. Ming), chengming_huang@hotmail.com (C. Huang).

http://dx.doi.org/10.1016/j.amc.2016.10.021
0096-3003/

Obviously, if we set $b(t) \equiv 0$ in Eq. (1.1), then it reduces to the following Volterra integral equation (VIE)

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \qquad t \in I. \tag{1.4}$$

The regularity of the exact solution, and the convergence properties of collocation solutions to (1.4) were well studied by Brunner (cf. [4], pp. 198–231). We will compare most of our results for Eq. (1.1) with the ones for Eq. (1.4) in the subsequent sections of this paper.

Eq. (1.1) can be obtained from the differentiated form of first-kind Volterra integral equation

$$(\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t) = g(t), \qquad t \in I. \tag{1.5}$$

In addition, if $K_2(t,s) = -K_1(t,s)$ on D_θ, VFIE (1.1) then reduces to

$$y(t) = g(t) + b(t)y(\theta(t)) + (\mathcal{W}_\theta y)(t), \qquad t \in I, \tag{1.6}$$

with

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)y(s)ds, \qquad t \in I.$$

This equation can be found in mathematical models in epidemiology, population growth and relevant phenomena in biology (cf. [16]).

There have been a lot of papers concerning numerical methods for *vanishing delay* Volterra integral equations and differential equations (cf. [2,3,6–13,15,18–20,23,24]). More recently, Xie et al. [21] studied the attainable order of convergence of collocation solutions on uniform meshes for the *vanishing delay* VFIE

$$y(t) = g(t) + b(t)y(\theta(t)) + \int_0^t K_1(t,s)y(s)ds + \int_0^{\theta(t)} K_2(t,s)y(s)ds, \qquad t \in [0,T], \tag{1.7}$$

where $\theta(0) = 0$. They found that the iterated collocation solution does not possess local superconvergence at the mesh points.

However, to the best of our knowledge, the numerical analysis for *non-vanishing delay* VFIE (1.1) with $b(t) \not\equiv 0$ has not yet been studied. The main difficulty lies in the fact that the representation of the collocation error, which will play a key role in the analysis of superconvergence of collocation solutions (cf. [5]), is not yet understood. It is the aim of the present paper to derive such a solution representation, and use it to analyze the impact of the delay term $b(t)y(\theta(t))$ on the convergence results for VFIE (1.1).

It will be shown in this paper that our results are not only a generalization of the corresponding results for (1.4), but also indicate that the delay term $b(t)y(\theta(t))$ will lead to

(i) a lower regularity of the exact solution to (1.1) at the primary discontinuity points;
(ii) a much more tedious deduction, and also a more complicated expression of the local representation of the exact solution to (1.1);
(iii) a reduction in the convergence order of collocation solutions to (1.1).

In addition, we will also show that collocation solutions may exhibit local superconvergence properties at certain non-mesh points.

This paper is organized as follows. In Section 2, we discuss the existence, uniqueness, regularity, and local representation of the exact solution to Eq. (1.1). Some (iterated) collocation approximations to (1.1) are constructed in Section 3. Section 4 focuses on the attainable (global and local) convergence order of collocation solutions. In Section 5, we present several numerical experiments to illustrate our theoretical results. Finally, some concluding remarks are given in Section 6.

2. Existence, uniqueness, regularity, and representation of the exact solution

We will assume that the delay function θ satisfies
(D1) $\theta(t) := t - \tau(t),\ \tau \in C^d(I)$ for some $d \geq 0$;
(D2) θ is strictly increasing on I;
(D3) $\tau(t) \geq \tau_0 > 0$ for $t \in I$.

It is well known that if the delay $\tau(t)$ satisfies condition (D3), then it will induce the so-called primary discontinuity points $\{\xi_\mu\}$, and these points can be generated by the recursion

$$\theta(\xi_{\mu+1}) = \xi_\mu, \qquad \mu = 0, 1, \ldots; \qquad \xi_0 := t_0. \tag{2.1}$$

Moreover, condition (D2) ensures that $\{\xi_\mu\}$ is strictly increasing. For simplicity and without loss of generality, we assume that $T = \xi_M$ (or, $\xi_M < T < \xi_{M+1}$ for some positive integer M).

Since solutions of non-vanishing delay problems generally suffer from a loss of regularity at $\{\xi_\mu\}$, we divide the whole interval I into several 'macro-intervals' $I^{(\mu)} := (\xi_\mu, \xi_{\mu+1}]$ $(0 \leq \mu \leq M-1)$ by $\{\xi_\mu\}$, and denote

$$Z_\mu := \left\{\xi_\mu : t_0 = \xi_0 < \xi_1 < \cdots < \xi_M = T, \quad \theta(\xi_{\mu+1}) = \xi_\mu\right\}.$$

as the set of these primary discontinuity points.

In order to study the contribution of the delay term $b(t)y(\theta(t))$ in (1.1) on some well-posedness of the exact solution, and also on the convergence results of collocation solutions, we will assume that $b(t) \not\equiv 0$ in Eq. (1.1) hereafter in this paper.