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Piecewise Extended Chebyshev spaces: A numerical test for design

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ABSTRACT

Given a number of Extended Chebyshev (EC) spaces on adjacent intervals, all of the same dimension, we join them via convenient connection matrices of the maximum order. The global space is called a Piecewise Extended Chebyshev (PEC) space. In such a space one can count the total number of zeroes of any non-zero element, exactly as in each EC-section-space. When this number is bounded above in the global space the same way as in its section-spaces, we say that it is an Extended Chebyshev Piecewise (ECP) space. A thorough study of ECP-spaces has been developed in the last two decades in relation to blossoms, with a view to design. In particular, extending a classical procedure for EC-spaces, ECP-spaces were recently proved to all be obtained by means of piecewise generalised derivatives. This yields an interesting constructive characterisation of ECP-spaces. Unfortunately, except for low dimensions and for very few adjacent intervals, this characterisation proved to be rather difficult to handle in practice. To try to overcome this difficulty, in the present article we show how to reinterpret the constructive characterisation as a theoretical procedure to determine whether or not a given PEC-space is an ECP-space. This procedure is then translated into a numerical test, whose usefulness is illustrated by relevant examples.

1. Introduction

By their ability to ensure unisolvence of Hermite interpolation problems or, equivalently, by the bound on the number of zeroes of their non-zero elements, Extended Chebyshev spaces are known as the most natural generalisations of polynomial spaces, and for this reason they are old tools in Approximation Theory [10,34]. In that direction they are generally defined by means of generalised derivatives associated with systems of weight functions, which permits to extend to them various well-known notions of the polynomial framework, *e.g.*, generalised divided differences [31] and associated Newton-type decompositions, Taylor formulæ, ...[12,31,34].

Initiated by Pottmann [33], the theory of Chebyshevian blossoming has permitted a deeper understanding of Extended Chebyshev spaces and Chebyshevian splines (*i.e.*, splines with pieces taken from the same Extended Chebyshev space and with ordinary continuity at the knots [2,3]), while enhancing their resemblance with polynomial (spline) spaces in connection with geometric design. The present paper is not at all a paper on blossoms, but it would not exist without the fundamental contribution of these powerful and elegant tools. In any situation where blossoms arise, the major difficulty

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consists in proving their pseudoaffinity in each variable which extends the well-known affinity in each variable of polynomial blossoms. Once this proven, the classical design algorithms are somehow inherent in Chebyshevian blossoms which also guarantee shape preservation of the resulting Bernstein bases [18]. As a recent important progress arising from blossoms, let us mention the complete description of all possible systems of weight functions which can be associated with a given Extended Chebyshev space on a closed bounded interval [25]. Moreover the geometrical nature of Chebyshevian blossoms makes them ideal tools to express geometric contact between parametric curves. This naturally produces blossoms for Chebyshevian splines with similar properties and consequences, *e.g.*, geometric design algorithms, B-spline bases, shape preservation.

Unlike polynomials, Extended Chebyshev spaces explicitly or implicitly involve shape parameters and this explains why they offer more possibilities in the control of shapes of curves and surfaces. To take full advantage of the Chebyshevian framework, it is useful to consider *Piecewise Chebyshevian splines*, that is, splines with pieces taken from different Extended Chebyshev spaces all of the same dimension, the continuity between consecutive pieces being controlled by connection matrices with orders specified by knot multiplicities as is usual. These splines were first considered by Barry in [1], see also [15,16,32] and [7,8] for geometrically continuous polynomial splines. To be of interest for applications, and in particular for geometric design, such a spline space $\mathbb{S}$ is expected to possess a B-spline basis – in the usual sense of a normalised basis composed of minimally supported splines – and this feature should be maintained after knot insertion. As a matter of fact, this requirement was proved to be equivalent to the existence of blossoms in the space $\mathbb{S}$ [17,19,24]. Moreover, for an efficient control of the shapes, the B-spline bases are additionally expected to be totally positive [9]. This property can automatically be derived from the properties of blossoms, and in particular from their pseudoaffinity. This explains why the terminology "*$\mathbb{S}$ is good for design*" was adopted whenever the spline space $\mathbb{S}$ possesses blossoms. It should be mentioned that the interest of Piecewise Chebyshevian spline spaces good for design is not limited to design: they also naturally produce multiresolution analyses with associated piecewise Chebyshevian wavelets [13], they permit approximation by Schoenberg-type operators [27], they have useful applications in Isogeometric Analysis [14], ...

In this article we focus on the special case where all interior knots have zero multiplicities. Then, the corresponding spline spaces have the same dimension as their section-spaces and they are referred to as *Piecewise Extended Chebyshev spaces* (PEC). The first motivation to consider this case is that zero multiplicities can efficiently be used to strengthen the shape effects [11]. The second motivation lies in the fact that determining the class of all piecewise Chebyshevian spline spaces which are good for design amounts to determining the class of all PEC-spaces which are good for design. Indeed, it was recently proved that a piecewise Chebyshevian spline space is good for design if and only if it is based on a PEC-space good for design, that is, possessing blossoms [19,24]. It is known that a given PEC-space $\mathbb{E}$ which contains constants possesses blossoms if and only if the PEC-space obtained from $\mathbb{E}$ by differentiation is an Extended Chebyshev piecewise space (ECP) in the sense that the global bound on the number of zeroes is exactly the same as in each of its section-spaces. The presence of blossoms in $\mathbb{E}$ can also be characterised by the existence of systems of weight functions associated with the section-spaces, relative to which the continuity conditions are expressed by identity matrices [23]. Given that we know how to obtain all possible systems of weight functions associated with the section-spaces, this characterisation naturally provides us with a procedure to determine whether or not a given PEC-space is good for design. Nevertheless, in general this procedure proves to be all the more difficult to carry out in practice as zero multiplicities allow no freedom between consecutive sections. This motivated the search for an effective numerical procedure as a replacement, to which the present work is devoted.

The paper is organised as follows. The necessary background is presented in Section 2, with special insistence on Bernstein and Bernstein-like bases and their behaviour under possible piecewise generalised derivatives associated with piecewise weight functions according to a process similar to the non-piecewise case. In Section 3, these results are first reinterpreted as a theoretical test to answer the question: *is a given PEC-space an ECP-space?*, which is in turn transformed into a numerical test. What we actually test is: can we repeatedly diminish the dimension via piecewise generalised derivatives? We illustrate this test by relevant examples in Section 4, in particular with a view to design with shape parameters. We conclude the paper with some comments on both the usefulness and the limits of the numerical procedure.

2. Background

In this section we briefly survey the main results on $(n+1)$-dimensional piecewise spaces obtained from $(n+1)$-dimensional section-spaces on adjacent intervals joined by connecting left/right derivatives at the interior knots by appropriate matrices. These results were proved in many earlier articles by the third author to which we refer the reader to, *e.g.*, [20,22,23,28] and other references therein. This survey is deliberately presented in a way to facilitate the next section.

2.1. Piecewise spaces via connection matrices

Throughout this article we consider a fixed interval $[a, b]$, $a < b$, and a fixed sequence $\mathbb{T} = (t_1, \dots, t_q)$ of $q \geq 1$ knots interior to $[a, b]$, with

$$t_0 := a < t_1 < \cdots < t_q < t_{q+1} := b.$$