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Center stable manifold for planar fractional damped equations

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ABSTRACT

In this paper, we discuss the existence of a center stable manifold for planar fractional damped equations. By constructing a suitable Lyapunov–Perron operator via giving asymptotic behavior of Mittag–Leffler function, we obtain an interesting center stable manifold theorem. Finally, an example is provided to illustrate the result.

1. Introduction

Fractional derivatives has been recognized as one of the best tools to describe long-memory processes. The corresponding mathematical models of these processes are fractional differential equations, which has been considered as an alternative model to integer differential equations. Recently, the subject of fractional differential equations is gaining much importance and attention. For more history and basic results on fractional calculus theory, one can see monograph [1–8] and the references therein.

In the past decades, various results for Cauchy problem, boundary value problem, nonlocal problem, impulsive problem, Ulam–Hyers stability, and control problem of Riemann–Liouville type, Caputo type, Hadamard type fractional differential equations or inclusions have been paid more and more attention in [9–28], our works [29–40] and other recent contributions [41–44]. However, the development of a stable manifold theory for nonlinear fractional differential equations is still in its infancy.

Very recently, Cong et al. [24] initial to present an interesting local stable manifold theorem near a hyperbolic equilibrium point for planar fractional differential equations of the order $\alpha \in (0, 1)$ by defining a related Lyapunov–Perron operator for two dimensional fractional systems and dealing with asymptotic behavior of Mittag–Leffler function $E_{\alpha,\alpha}$. Then, the fixed point of Lyapunov–Perron operator describes the set of all solutions near the fixed point tending to zero, which is called the stable manifold of the hyperbolic fixed point.

In [45], Schäfer and Kempfle pointed that the classical damped model $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t),\ a, b \in \mathbb{R}^+$ cannot be suitable to characterize the effects of hysteretic or viscoelastic damping. The authors [45] offer a new fractional damped model (damped equations with fractional order derivative) by changing the first-order derivative into a fractional case:

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$\ddot{x}(t) + a^c D_t^\beta x(t) + bx(t) = f(t),\ \ a, b \in \mathbb{R}^+,\ \ \beta \in (0, 2)$, which gives a very good model for characterizing the damping behavior of viscoelastic materials verified by experiments [46]. Thus, it is natural to change the second-order derivative into a differential fractional case to obtain a more generalized fractional damped equation: $a^c D_t^\alpha + a^c D_t^\beta x(t) + bx(t) = f(t),\ \ a, b \in \mathbb{R}^+,\ \ \alpha, \beta \in (0, 2)$. This is a crucial motivation to study our problem (1).

In this paper, we prove a center stable manifold theorem for planar fractional damped equations involving two differential Caputo derivatives. We develop and extend the idea and methods in [24] focusing on asymptotic behavior of two different parameter Mittag–Leffler functions $E_{\alpha,\beta}$, $\alpha \in (1, 2)$, $\beta \in (0, 1)$, which will be used to obtain center stable manifolds result.

The original contributions of this paper are stated as follows:

(i) We establish a framework to deal with local center stable manifold for planar damped equations with two different Caputo fractional derivatives, which extend the classical planar damped equations with first order and second order derivatives. In particular, a suitable Lyapunov–Perron operator is constructed.
(ii) We give some useful estimations of integral inequalities involving two parameter Mittag–Leffler functions. These techniques and results can be used to deal with stability problems for several types of fractional differential equations.
(iii) A special numerical example demonstrates our theoretical result via a figure.

The rest of this paper is organized as follows. In Section 2, we give some results for asymptotic behavior of Mittag–Leffler functions $E_{\alpha,\beta}$. In Section 3, we recall some fundamental results on fractional calculus and fractional damped equations involving two differential Caputo derivatives, and set our problem of the study. The final Section 4 is devoted to the main result of this paper about center stable manifolds and an example is given to demonstrate the application of our main result.

2. Asymptotic behavior of Mittag–Leffler functions $E_{\alpha,\beta}$

The two parameter Mittag–Leffler functions $E_{\alpha,\beta}(z)$ are described as:

$$E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha + \beta)},$$

$z \in \mathbb{R}$ and α, β are positive real numbers. Next, $E_\alpha(z) = E_{\alpha,1}(z)$.

To give some results for asymptotic behavior of Mittag–Leffler functions $E_{\alpha-\beta,2}$ and $E_{\alpha-\beta,\alpha}$ for $\alpha \in (1, 2)$ and $\beta \in (0, 1)$, we recall the following

Lemma 2.1. *(see [47]) Let $\bar{\alpha} \in (0, 2)$ and $\bar{\beta} \in \mathbb{R}$ be arbitrary. Then for $\bar{p} = [\frac{\bar{\beta}}{\bar{\alpha}}]$, the following asymptotic expansions hold:*
(i)

$$E_{\bar{\alpha},\bar{\beta}}(z) = \frac{1}{\bar{\alpha}} z^{\frac{1-\bar{\beta}}{\bar{\alpha}}} \exp(z^{\frac{1}{\bar{\alpha}}}) - \sum_{k=1}^{\bar{p}} \frac{z^{-k}}{\Gamma(\bar{\beta} - \bar{\alpha}k)} + O(z^{-1-\bar{p}}) \quad \text{as } z \to \infty.$$

(ii)

$$E_{\bar{\alpha},\bar{\beta}}(z) = -\sum_{k=1}^{\bar{p}} \frac{z^{-k}}{\Gamma(\bar{\beta} - \bar{\alpha}k)} + O(|z|^{-1-\bar{p}}) \quad \text{as } z \to -\infty.$$

By inserting $\bar{\alpha} = \alpha - \beta,\ \bar{\beta} = 2$ and $z = t^{\alpha-\beta}\lambda$, we give the first asymptotic expansions for Mittag–Leffler functions which extend [24, Lemma 3] to our case.

Lemma 2.2. *For any $\lambda > 0$, $\alpha \in (1, 2)$, $\beta \in (0, 1)$ and $p = [\frac{2}{\alpha-\beta}]$, it holds*
(i)

$$tE_{\alpha-\beta,2}(\lambda t^{\alpha-\beta}) = \frac{\exp(\lambda^{\frac{1}{\alpha-\beta}} t)}{\lambda^{\frac{1}{\alpha-\beta}}(\alpha-\beta)} - \sum_{k=1}^{p} \frac{t^{1-k(\alpha-\beta)}}{\lambda^k \Gamma(2 - (\alpha-\beta)k)} + O(t^{1-(\alpha-\beta)(1+p)}) \quad t \to \infty.$$

(ii)

$$tE_{\alpha-\beta,2}(-\lambda t^{\alpha-\beta}) = -\sum_{k=1}^{p} \frac{t^{1-k(\alpha-\beta)}}{(-\lambda)^k \Gamma(2 - (\alpha-\beta)k)} + O(t^{1-(\alpha-\beta)(1+p)}) \quad t \to \infty.$$

By inserting $\bar{\alpha} = \alpha - \beta,\ \bar{\beta} = \alpha$ and $z = t^{\alpha-\beta}\lambda$, we give asymptotic expansions for Mittag–Leffler functions $E_{\alpha-\beta,\alpha}$.

Lemma 2.3. *For any $\lambda > 0$, $\alpha \in (1, 2)$ and $\beta \in (0, 1)$ with $p = 1$, it holds*