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Cyclicity of some analytic maps

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ABSTRACT

In this paper, we describe an approach to estimate the cyclicity of centers in maps given by $f(x) = -x - \sum_{k=1}^{\infty} a_k x^{k+1}$. The main motivation for this problem originates from the study of cyclicity of planar systems of ODEs. We also consider the bifurcation of limit cycles from each component of the center variety of some particular cases of maps $f(x) = -x - \sum_{k=1}^{\infty} a_k x^{k+1}$ arising from algebraic equations of the form $x + y + h.o.t. = 0$ where higher order terms up to degree four are present.

1. Introduction

In the qualitative theory of planar polynomial differential systems there are two problems that have been extensively considered: Hilbert's sixteenth problem, which asks for the number and position of limit cycles in polynomial planar vector fields(see for instance [9,11] and references therein) and the center-focus problem, which aims to distinguish between a center and a focus (see for instance [1,6,16] and references given there). These two problems are strongly related. For instance, the study of the center conditions gives rise to the Lyapunov constants that are often used to produce limit cycles. This paper deals with the study of cyclicity of periodic points of iterated maps

$$f(x) = -\sum_{k=0}^{\infty} a_k x^{k+1}, \tag{1.1}$$

in a neighborhood of the origin ($x = 0$), where f is a real function of one real variable, i. e. $x \in \mathbb{R}$, $a_k \in \mathbb{R}$ for all k and $a_0 = 1$ and $y = f(x)$ is the solution to the algebraic equation $x + y + \sum_{i+j=2}^{n} \alpha_{ij} x^i y^j$. This problem was proposed in [19].

In [3] the author proposes an approach to estimate the cyclicity of center of some classes of planar polynomial systems of the form

$$\begin{aligned} \dot{x} &= -y + P(x, y) \\ \dot{y} &= x + Q(x, y), \end{aligned} \tag{1.2}$$

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where P and Q are polynomials beginning with quadratic terms. Roughly speaking the cyclicity can be found from examining the coefficients and common zeros of the return map associated to the vector field corresponding to (1.2).

In this paper, we adopt the approach described in [3] to discrete dynamical systems with centers and apply it to some particular classes of maps. Basically, we follow the Bautin idea [2] and the references [3,14,15,17].

Analogously to (1.2) we can define a monodromic point for maps given by (1.1) and consider the concepts of stable and unstable foci and center points for such maps. Using this analogy one can consider the return and difference maps for applications of the form (1.1) and obtain their Lyapunov coefficients. Each one of these concepts are defined in detail in the following section. In order to make the paper more self-contained we summarize the theory of computing the focus quantities presented in [14]. In Section 3 we adopt the technique which was introduced for continuous systems in [3] in order to obtain the main result of this paper, Theorem 1. In section four we apply this theorem to some new examples partially using the results from [14,17]. In the last section we present some conclusions.

2. Maps and limit cycle bifurcations

In this section, we summarize some important definitions and results concerning the dynamics of maps given by (1.1). For more details see [14,17].

For map (1.1) we denote by f^p $(p \in \mathbb{N})$ the pth iterate of the map.

Definition 1. The singular point $x = 0$ of the map (1.1) is called

(1) *a stable focus*, if there exists an $\varepsilon > 0$ such that for all x for which $|x| < \varepsilon$ we have $\lim_{k\to\infty} f^k(x) = 0$,
(2) *an unstable focus*, if it is a stable focus for the (inverse) map f^{-1},
(3) *a center*, if there exists an $\varepsilon > 0$ such that for all x for which $|z| < \varepsilon$ the equality $f^2(x) = x$ holds.

A point $x_0 > 0$ is called a *limit cycle* of the map (1.1) if x_0 is an *isolated root* of the equation

$$f^2(x) - x = 0. \tag{2.1}$$

Consider the equation

$$\Psi(x, y) = x + y + \sum_{i+j=2}^{n} \alpha_{ij} x^i y^j = 0, \tag{2.2}$$

where α_{ij}, x, y are real. For each fixed n this equation has a unique analytic solution of the form (1.1). From the study of solutions to (2.2) we have the following problem: how to find in the space of coefficients $\{\alpha_{ij}\}$ the manifold for which the corresponding map f has a center at the origin and to investigate the bifurcation of limit cycles of such a map. This problem was firstly stated in [19]. The center-focus problem and the problem of estimating the number of limit cycles near $x = 0$ for map given by (1.1) have many similarities to the corresponding problem stated for planar quadratic differential systems, so that the terminology introduced above originates from the qualitative theory of planar differential systems.

In order to study the number of limit cycles near $x = 0$ we will use the approach based on Lyapunov function, Φ, as done in [14,19]. If Φ is a Lyapunov function of the map (1.1) then

$$\Phi(f(x)) - \Phi(x) = g_2 x^4 + g_4 x^6 + \cdots + g_{2k} x^{2k+2} + \cdots \tag{2.3}$$

and the coefficients g_{2k} from (2.3) are called *focus quantities*. Its formal series is given by

$$\Phi(x) = x^2 \left(1 + \sum_{k=1}^{\infty} b_k x^k \right). \tag{2.4}$$

Remark 1. The authors in [14] proved that if $g_{2k} = 0$ for all $k \in \mathbb{N}$ then the map (1.1) has a center in the origin. If $g_2 = \cdots = g_{2k-2} = 0$ and $g_{2k} \neq 0$ then $x = 0$ is a stable (respectively, unstable) focus when $g_{2k} < 0$ (respectively, $g_{2k} > 0$) for map (1.1). This actually confirms that the Lyapunov function (2.4) is a concept analogous to the concept of Lyapunov function for planar systems of ODEs.

In Figs. 1 and 2 the geometric meaning of the dynamics mentioned in Definition 1 for system (1.1)) is presented. In Figs. 1 and 2 examples of a center and a stable focus, respectively, are shown.

Analogously to the case of planar ODEs we can define the Poincaré return map

$$\mathrm{R}(x) = f^2(x) = x + c_2 x^3 + c_3 x^4 + \cdots$$

and the difference map

$$\mathrm{P}(x) = f^2(x) - x = c_2 x^3 + c_3 x^4 + \cdots. \tag{2.5}$$

Moreover, as a limit cycle is an isolated (positive) root of equation $f^2(x) - x = 0$ and there is a one to one correspondence between $x_0 > 0$ and $f(x_0) < 0$, if x_0 is a limit cycle of f then one must consider also the image point of x_0 by f which