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Analysis of splitting methods for solving a partial integro-differential Fokker–Planck equation☆

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ABSTRACT

A splitting implicit-explicit (SIMEX) scheme for solving a partial integro-differential Fokker–Planck equation related to a jump-diffusion process is investigated. This scheme combines the Chang–Cooper method for spatial discretization with the Strang–Marchuk splitting and first- and second-order time discretization methods. It is proved that the SIMEX scheme is second-order accurate, positive preserving, and conservative. Results of numerical experiments that validate the theoretical results are presented.

1. Introduction

The Fokker–Planck (FP) equation governs the time evolution of the probability density function (PDF) of stochastic processes and plays a fundamental role in many problems involving random quantities [14,23,24,30]. The FP equation has been first applied to problems with randomness given by Brownian motion; in this case, the derivation of the FP equation, some methods of solution and its application to diffusion models can be found in [10,14,28,30].

In the last decade, one can see a growing interest in stochastic processes with jumps. This class of problems includes Lévy processes, whose increasing popularity stems, e.g., from the need of modeling the market behavior beyond the Black–Scholes framework [8].

In this work, we consider a jump-diffusion Markov process $X_t \in \mathbb{R}^d$, for $t \in I = [t_0, t_f]$, that solves the following stochastic initial value problem

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t + dP_t, \\ X_{t_0} = X_0, \end{cases} \tag{1}$$

where $X_0 \in \mathbb{R}^d$ is a given initial random data. This stochastic differential equation (SDE) relates the infinitesimal increments of the stochastic process X_t to both deterministic and random increments, given by the multidimensional Wiener process

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$W_t \in \mathbb{R}^m$ and the compound Poisson process $P_t \in \mathbb{R}^d$. We denote with $\lambda \in \mathbb{R}^+$ the rate of the time events of the compound Poisson process and with $g(y)$ the PDF of the size of its jumps. The density $g(y)$ is nonnegative and normalized, $\int_{\mathbb{R}^d} g(y)dy = 1$. The deterministic functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ represent the drift and the diffusion coefficients, respectively. We assume that the matrix σ is full-rank. The solvability of (1) follows under growth and regularity conditions on b and σ; see, e.g., [3,21].

A powerful tool to examine the stochastic problem (1) is the investigation of the time dependent PDF $f(x, t)$ of X_t, because it characterizes the statistics of the process over the time interval I. The PDF of X_t is defined in $\mathbb{R}^d \times I$ and it is governed by the following partial integro-differential equation (PIDE) of FP type

$$\partial_t f(x,t) = \mathcal{L} f(x,t) + \mathcal{I} f(x,t), \tag{2}$$

where the two linear operators $\mathcal{L}$ and $\mathcal{I}$ are defined as follows

$$\begin{aligned} \mathcal{L} f(x,t) &:= -\sum_{i=1}^{d} \partial_{x_i}(b_i(x) f(x,t)) + \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\left(a_{ij}(x) f(x,t)\right) \\ \mathcal{I} f(x,t) &:= \lambda \int_{\mathbb{R}^d} f(y,t) g(x-y) dy - \lambda f(x,t), \end{aligned} \tag{3}$$

where $a_{ij}(x)$ are the elements of a matrix defined as $a_{ij}(x) := \frac{1}{2}\sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x)$. Since the diffusion coefficient σ is full rank, the matrix a is positive definite. For more details about the link between PIDEs of FP type and Markov processes, see [10,14,32]. From a probabilistic point of view, integro-differential operators are closely linked to the concept of group generator of a stochastic process; for a detailed discussion about the generator of a Lévy process, see [3].

Classical solutions of initial-boundary problems containing PIDEs have been examined in the context of Hölder spaces and uniformly parabolic operators [15–17]. More in general, existence of solutions is considered in the framework of viscosity solutions [11]. Numerical schemes for viscosity solutions of PIDEs are discussed, e.g., in [4] and references therein.

In financial applications, PIDEs naturally arise when option prices in jump diffusion models have to be computed [1,5,8,9,27]. An overview on several numerical schemes for PIDEs arising in the pricing problem of financial derivatives can be found in [13]. However, a rigorous numerical analysis of these schemes is often not available.

In this work, we consider an initial-boundary value problem governed by (2)–(3) with the aim of computing the PDF of X_t. Since $f(x, t)$ is the PDF of a stochastic process, it must be positive and its integral over the evolution domain of the process must be equal to 1. These two properties hold for the continuous problem endowed with suitable initial and boundary conditions [14,32]; we require that these two structural properties have to be valid also for the discretized FP equation.

With the aim of solving the FP equation related to a ionized plasma, Chang and Cooper [7] proposed a conservative finite difference scheme that turns out to be one of the most appropriate schemes for discretizing the FP spatial operator. A complete numerical analysis of this scheme has been carried out in [26]. Conservative numerical schemes for our FP problem in case of a jump-diffusion process have been less investigated.

The purpose of our work is to provide a numerical analysis of a conservative scheme that solves the FP problem for a jump-diffusion process, with appropriate initial and boundary conditions. We consider the framework of the method of lines (MOL) [6,31]. The differential operator $\mathcal{L}$ in (3) is discretized with the Chang–Cooper (CC) scheme, and the integral operator $\mathcal{I}$ in (3) is approximated by a quadrature formula. This leads to a large system of ordinary differential equations, that we approximate with the the Strang–Marchuk (SM) splitting method [18,19,25,33]. The SM method decomposes the original problem in a sequence of different sub-problems with simpler structure, which are separately solved and linked to each other through initial conditions and final solutions. A splitting method allows to solve each sub-problem implicitly or explicitly, depending on the nature of the sub-problem. After performing the SM splitting, we carry out the time integration with a first- and a second-order time-differencing method. Our discretization procedure with the two different time-discretization schemes leads to the SIMEX1 and SIMEX2 schemes, respectively. For clarity, our discretization workflow is summarized in Fig. 1. We remark that splitting methods and finite differences are frequently used by practitioners [13]. However, less attention has been put on positivity and conservation properties, and the numerical analysis has focused mainly on time-approximation properties. In this paper, we prove that the combination of the CC scheme with the SM method results in accurate discretization schemes that guarantee conservativeness of the total probability and non-negativity of the PDF solutions.

This paper is organized as follows. The next section defines our FP problem, including a source term for analysis purposes. Correspondingly, we investigate existence and uniqueness of solutions to this problem in the case of unbounded and bounded domains. In Section 3, we illustrate the CC scheme and the SM splitting method, and their combination for constructing our SIMEX schemes. Section 4 is devoted to the convergence analysis of our SIMEX1 and SIMEX2 schemes. For these schemes, we prove stability in time and second-order accuracy in space. Further, we prove first- and second-order accuracy in time for the SIMEX1 and SIMEX2 schemes, respectively. In Section 5, we prove that our numerical schemes guarantee non-negativity and conservation of total probability of the PDF solution. Section 6 presents results of numerical experiments that validate our theoretical results. The Appendix illustrates a possible choice for the truncation of the domain of definition of the Fokker–Planck problem. A section of conclusion completes this work.