Applied Mathematics and Computation 294 (2017) 77–86

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On approximation properties of Baskakov–Szász–Stancu operators using hypergeometric representation

Vishnu Narayan Mishra [a,b,*], Ram N. Mohapatra [c], Preeti Sharma [a]

[a] *Applied Mathematics & Humanities Department, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat 395 007, Gujarat, India*
[b] *L. 1627 Awadh Puri Colony Beniganj, Phase -III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad 224 001, Uttar Pradesh, India*
[c] *University of Central Florida, Orlando, FL 32816, USA*

ARTICLE INFO

MSC:
41A25
41A36

Keywords:
Hypergeometric series
Baskakov–Szász–Stancu operator
Modulus of continuity
Simultaneous approximation
Degree of approximation

ABSTRACT

In the present paper, we study the mixed summation integral type operators having Baskakov and Szász basis functions in summation and integration form, respectively. We also give here the alternate form of the operators in terms of hypergeometric functions and estimated moments of these operators using hypergeometric series. In the last section, we present some results for operators related to convergence.

1. Introduction

In 1993 Gupta–Srivastava [5] introduced the mixed Durrmeyer type operators by taking the weight functions as the Szász basis function for any $f \in C[0, \infty)$ as

$$\mathcal{L}_n(f, x) = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t) dt, \quad x \in [0, \infty), \tag{1.1}$$

and $p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$ and $s_{n,k}(t) = e^{-nt} \frac{(nt)^k}{k!}$.

It is observed from [5] that these operators reproduce only the constant functions. Recently, Mishra and Sharma [9] introduced Stancu type generalization of operators (1.1) as

$$\mathcal{L}_n^{(\alpha,\beta)}(f, x) = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f\left(\frac{nt + \alpha}{n + \beta}\right) dt, \quad x \in [0, \infty). \tag{1.2}$$

One can write it as

$$\int_0^{\infty} W_n(x, t) f\left(\frac{nt + \alpha}{n + \beta}\right) dt$$

* Corresponding author. Tel. +91 99133 87604.
E-mail addresses: vishnunarayanmishra@gmail.com, vishnu_narayanmishra@yahoo.co.in, v_n_mishra_hifi@yahoo.co.in, vishnu_narayanmishra@rediffmail.com (V.N. Mishra), ram.mohapatra@ucf.edu (R.N. Mohapatra), preeti.iitan@gmail.com (P. Sharma).

http://dx.doi.org/10.1016/j.amc.2016.08.049
0096-3003/

where

$$W_n(x,t) = n\sum_{k=0}^{\infty} p_{n,k}(x)s_{n,k}(t).$$

Many operators have been built and studied by using Baskakov weight functions mixed with other weight functions. Here we refer to Srivastava [14], Mishra et al. [9–12], Singh et al. [1] and Icoz and Mohapatra [7], Gairola et al. [4], Büyükyazici [2,3], Wafi et al. [15], Mishra et al. [16–21] and references therein. Gupta and Srivastava [5] estimated direct results in simultaneous approximation and established an asymptotic formula and estimation of error. In the year 1991 Kasana et al. [8] find some approximation results for Baskakov operators. Here we present alternate form of such operators and find the moments using hypergeometric representation. We may rewrite these operators (1.1) in the form of hypergeometric function as

$$\begin{aligned}\mathcal{L}_n(f,x) &= n\sum_{k=0}^{\infty}\frac{\Gamma(n+k)}{\Gamma(n)k!}\frac{x^k}{(1+x)^{n+k}}\int_0^\infty \frac{e^{-nt}(nt)^k}{k!}f(t)dt\\ &= \frac{n}{\Gamma(n)(1+x)^n}\int_0^\infty e^{-nt}f(t)dt\sum_{k=0}^{\infty}\frac{\Gamma(n+k)}{(k!)^2}\frac{(nxt)^k}{[(1+x)]^k}.\end{aligned}$$

Using the identity $\Gamma(n+k)=(n)_k\Gamma(n)$ and $(1)_k=k!$ we have

$$\mathcal{L}_n(f,x) = \frac{n}{\Gamma(n)(1+x)^n}\int_0^\infty e^{-nt}f(t)dt\sum_{k=0}^{\infty}\frac{\Gamma(n)(n)_k}{(k!)(1)_k}\frac{(nxt)^k}{[(1+x)]^k}.$$

By hypergeometric series ${}_1F_1(a;b;x)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k k!}x^k$, we can write

$$\mathcal{L}_n(f,x) = \frac{n}{(1+x)^n}\int_0^\infty e^{-nt}f(t)\ \ {}_1F_1\left(n;1;\frac{nxt}{(1+x)}\right)dt.$$

This is the another form of the operators (1.1) in terms of the hypergeometric functions.

For α and β be two given real parameters satisfying the conditions $0\le\alpha\le\beta$ and for $f\in L_1[0,\infty)$, the Baskakov–Szász–Stancu operators are defined as

$$\mathcal{L}_n^{(\alpha,\beta)}(f,x) = \frac{n}{(1+x)^n}\int_0^\infty e^{-nt}f\left(\frac{nt+\alpha}{n+\beta}\right)\ \ {}_1F_1\left(n;1;\frac{nxt}{(1+x)}\right)dt.$$

2. Preliminary results

In this section, we give the following Lemma in terms of hypergeometric representation, which facilitates calculating moments of higher order.

Lemma 1 [6]**.** *If $e_r=t^r$, $r=0,1,2.$ we have*

$$\mathcal{L}_n(t^r,x) = \frac{x\Gamma(r+1)}{n^{r-1}}{}_2F_1\Big(n+1,1-r;2;-x\Big). \tag{2.1}$$

By simple computation from Lemma 1, we have

$$\mathcal{L}_n(1,x)=1,\ \ \mathcal{L}_n(t,x)=\frac{nx+1}{n},$$

$$\mathcal{L}_n(t^2,x)=\frac{1}{n^2}[n(n+1)x^2+4nx+2].$$

Lemma 2 [9]**.** *The following equalities hold for operator $\mathcal{L}_n^{(\alpha,\beta)}$:*

$$\begin{aligned}\mathcal{L}_n^{(\alpha,\beta)}(1,x) &= 1,\\ \mathcal{L}_n^{(\alpha,\beta)}(t,x) &= \frac{nx+1+\alpha}{n+\beta},\\ \mathcal{L}_n^{(\alpha,\beta)}(t^2,x) &= \frac{n(n+1)x^2}{(n+\beta)^2}+\frac{(4n+2n\alpha)x}{(n+\beta)^2}+\frac{(2+2\alpha+\alpha^2)}{(n+\beta)^2}.\\ \mathcal{L}_n^{(\alpha,\beta)}(t^r,x) &= x^r\frac{1}{(n+\beta)^r}\frac{(n+r-1)!}{(n-1)!}\\ &\quad +x^{r-1}\left\{\frac{r^2}{(n+\beta)^r}\frac{(n+r-2)!}{(n-1)!}+r\alpha\frac{1}{(n+\beta)^r}\frac{(n+r-2)!}{(n-1)!}\right\}\end{aligned}$$