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Series associated to some expressions involving the volume of the unit ball and applications

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ABSTRACT

The aim of this paper is to construct asymptotic series associated to some expressions involving the volume of the n-dimensional unit ball. New refinements and sharpenings of some old and recent inequalities on the volume of the n-dimensional unit ball are presented.

1. Introduction and motivation

In the recent past, inequalities about the volume of the unit ball in $\mathbb{R}^n$:

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \qquad (n \in \mathbb{N}) \tag{1}$$

have attracted the attention of many authors. See, *e.g.*, [2–23], where also further results on the gamma function are obtained. Here Γ denotes Euler's gamma function defined for every real number $x > 0$, by the formula:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt,$$

while $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A wide series of research methods were imagined, ranging from convexity, monotonicity, Mean Value Theorems to several numerical methods, such as those using computer softwares for symbolic computation. It is true that these methods permit us to obtain several estimates of nice shape, but of limited accuracy.

We propose in this paper an original approach using the theory of asymptotic series. Let

$$f(n) \sim g(n) \qquad (n \in \mathbb{N}) \tag{2}$$

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be an approximation formula, in the sense that the ratio $\frac{f(n)}{g(n)}$ converges to 1, when n approaches infinity. In asymptotic theory, there is a tendency to improve (2) by adjoining new factors, possibly an entire series of the form

$$f(n) \sim g(n) \exp\left\{\sum_{j=1}^{\infty} \frac{\omega_j}{n^j}\right\}.$$

Truncated at the mth term, this series, also called *an asymptotic series*, provides approximations to any accuracy of $n^{-(m+1)}$, for every integer $m \geq 1$. Usually, it can be proven that these truncations are under-, or upper- approximations of the given function $f(n)$.

Such a phenomenon appears in the case of Stirling's formula

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \qquad (x \in \mathbb{R};\ x \to \infty)$$

and its associated asymptotic series, as $x \to \infty$:

$$\ln\Gamma(x+1) \sim \frac{1}{2}\ln 2\pi + \left(x+\frac{1}{2}\right)\ln x - x + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}. \tag{3}$$

Here B_j's are *Bernoulli's numbers*, defined by the generating function:

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}.$$

The first Bernoulli's numbers are $B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}$, while $B_{2k+1} = 0$, for every $k \in \mathbb{N}$. Replacement of these values in (3), yields

$$\ln\Gamma(x+1) \sim \frac{1}{2}\ln 2\pi + \left(x+\frac{1}{2}\right)\ln x - x + \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360360x^{11}} + \frac{1}{156x^{13}} - \cdots\right).$$

Other details about Stirling's formula and Bernoulli's numbers can be read in the basic monograph [1].

Alzer [2, Theorem 8] investigated the complete monotonicity of some functions involving the series (3), then he proved the following double inequality:

$$\alpha_{4p+3}(x) < \ln\Gamma(x+1) < \alpha_{4q+1}(x) \qquad (x > 0;\ p, q \in \mathbb{N}_0), \tag{4}$$

where in general, $\alpha_{2m-1}(x)$ denotes the truncation of the series (3) up to the term containing $x^{-(2m-1)}$:

$$\alpha_{2m-1}(x) = \frac{1}{2}\ln 2\pi + \left(x+\frac{1}{2}\right)\ln x - x + \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \qquad (x > 0;\ m \in \mathbb{N}).$$

Our contribution in this paper is the construction of the asymptotic series for the quantities

$$\Omega_n,\quad \frac{1}{n}\ln\Omega_n,\quad \Omega_n^{1/n},\quad \ln\frac{\Omega_{n-1}}{\Omega_n},\quad \frac{\Omega_{n-1}}{\Omega_n},\quad \ln\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}},\quad \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$$

and for other related functions. By truncation of these series, we obtain several lower- and upper- bounds for the functions involved. As example of the applicability of our products, we show how can be improved the following classical results:

- Chen and Lin [10] ($a = \frac{e}{2} - 1,\ b = \frac{1}{3}$):

$$\frac{1}{\sqrt{\pi(n+a)}}\left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \leq \Omega_n < \frac{1}{\sqrt{\pi(n+b)}}\left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \qquad (n \in \mathbb{N});$$

- Borgwardt [8] ($a = 0,\ b = 1$), Alzer [3] and Qiu and Vuorinen [22] ($a = \frac{1}{2},\ b = \frac{\pi}{2} - 1$):

$$\sqrt{\frac{n+a}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b}{2\pi}} \qquad (n \in \mathbb{N});$$

- Alzer [3] ($\alpha^* = \frac{3\pi\sqrt{2}}{4\pi+6},\ \beta^* = \sqrt{2\pi}$):

$$\frac{\alpha^*}{\sqrt{n}} \leq \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \frac{\beta^*}{\sqrt{n}} \qquad (n \in \mathbb{N});$$