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# Convergence rate results for steepest descent type method for nonlinear ill-posed equations

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ABSTRACT

Convergence rate result for a modified steepest descent method and a modified minimal error method for the solution of nonlinear ill-posed operator equation have been proved with noisy data. To our knowledge, convergence rate result for the steepest descent method and minimal error method with noisy data are not known. We provide a convergence rate results for these methods with noisy data. The result in this paper are obtained under less computational cost when compared to the steepest descent method and minimal error method. We present an academic example which satisfies the assumptions of this paper.



## 1. Introduction

Steepest descent method is used extensively (see [1–8] for linear ill-posed equations and [11,13] for nonlinear ill-posed equations) for solving ill-posed operator equations. In this study we consider a modified steepest descent method and a modified minimal error method for approximately solving the operator equation

$$F(x) = y, \tag{1.1}$$

where $F$: $D(F) \subseteq X \to Y$ is a nonlinear Fréchet differentiable operator between the Hilbert spaces $X$ and $Y$. Let $D(F)$, $\langle ., .\rangle$ and $\|.\|$, respectively stand for the domain of $F$, inner product and norm which can always be identified from the context in which they appear. Fréchet derivative of $F$ is denoted by $F'(.)$ and its adjoint by $F'(.)^*$. Further we assume that Eq. (1.1) has a solution $\hat{x}$, which is not depending continuously on the right-hand side data $y$, i.e., (1.1) is ill-posed.

It is assumed further that we have only approximate data $y^\delta \in Y$ with

$$\|y - y^\delta\| \leq \delta.$$

Steepest descent method was considered by Scherzer [13], Neubauer and Scherzer [11] for approximately solving (1.1). In general, steepest-descent method for (1.1) can be written as

$$x_{k+1} = x_k + \alpha_k s_k, \tag{1.2}$$

where $s_k$ is the search direction taken as the negative gradient of the minimization functional involved and $\alpha_k$ is the descent. For solving Eq. (1.1) with $y^\delta$ in place of $y$, method (1.2) was studied by Scherzer [13] when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$ and $\alpha_k = \frac{\|s_k\|^2}{\|F'(x_k)^* s_k\|^2}$. For linear operator $F$, Gilyazov [10] studied ($\alpha$-process) method (1.2) when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$

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and $\alpha_k = \frac{\langle (F^*F)^\alpha s_k, s_k\rangle}{\langle (F^*F)^\alpha s_k, F^*Fs_k\rangle}$. Vasin [14] considered a regularized version of the steepest descent method in which $s_k = -F'(x_k)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)$ and $\alpha_k = \frac{\|s_k\|^2}{\|F'(x_k)s_k\|^2 + \alpha\|s_k\|^2}$. Here and below $x_0$ is the initial guess. Also, observe that the TIGRA-method of Ramlau [12] is of the form (1.2) with $s_k = -[F'(x_k)^*(F(x_k) - y^\delta) + \alpha_k(x_0 - x_k)]$ and $\alpha_k = \beta_k$. Note that, in all these methods, one has to compute Fréchet derivative of $F$ at each iterate $x_k$ in each iteration step which is in general very expensive.

### 1.1. Preliminaries

Let $B(x, r)$, $\overline{B}(x, r)$ stand, respectively for the open and closed balls in $X$, with center $x \in X$ and of radius $r > 0$.

In [11], Neubauer and Scherzer considered the steepest descent method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_k)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_k)s_k\|^2} \end{aligned} \tag{1.3}$$

and the minimal error method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_k)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|s_k\|^2} \end{aligned} \tag{1.4}$$

in the noise free case and obtained the rate

$$\|x_k - \hat{x}\| = O(k^{-\frac{1}{2}}) \tag{1.5}$$

under the assumptions $(\mathcal{A})$:

$(\mathcal{A}_1)$ $F$ has a Lipschitz continuous Fréchet derivative $F'(.)$ in a neighborhood of $x_0$.

$(\mathcal{A}_2)$ $F'(x) = R_x F'(\hat{x})$, $x \in B(x_0, \rho)$ where $\{R_x \colon x \in B(x_0, \rho)\}$ is a family of bounded linear operators $R_x : Y \longrightarrow Y$ with

$$\|R_x - I\| \le C\|x - \hat{x}\|$$

where $C$ is a positive constant and

$(\mathcal{A}_3)$

$$x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^{\frac{1}{2}} z$$

for some $z \in X$.

In the present paper, we consider a modified steepest descent method and a modified minimal error method, in the case of noise free case, defined by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|s_k\|^2}{\|F'(x_0)s_k\|^2} \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k s_k \quad (k = 0, 1, 2, \ldots) \\ s_k &= -F'(x_0)^*(F(x_k) - y) \\ \alpha_k &= \frac{\|F(x_k) - y\|^2}{\|s_k\|^2}, \end{aligned} \tag{1.7}$$

respectively. Instead of assumptions $(\mathcal{A})$, we use the following assumptions $(\mathcal{C})$:

$(\mathcal{C}_0)$ $\|F'(x)\| \le m$ for some $m > 0$ and for all $x \in D(F)$.

$(\mathcal{C}_1)$ $F'(\hat{x}) = F'(x_0)G(\hat{x}, x_0)$ where $G(\hat{x}, x_0)$ is a bounded linear operator from $X \longrightarrow X$ with

$$\|G(\hat{x}, x_0) - I\| \le C_0\rho$$

where $C_0$ is a positive constant and $\rho \ge \|x_0 - \hat{x}\|$.

$(\mathcal{C}_2)$ $F'(x) = R(x, y)F'(y)$ $(x, y \in B(x_0, \rho))$ where $\{R(x, y) \colon x, y \in B(x_0, \rho)\}$ is a family of bounded linear operators $R(x, y) : Y \longrightarrow Y$ with

$$\|R(x, y) - I\| \le C_1\|x - y\|$$

for some positive constant $C_1$ and