Applied Mathematics and Computation 294 (2017) 180–194

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Representation of solutions of delayed difference equations with linear parts given by pairwise permutable matrices via $\mathcal{Z}$-transform

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ARTICLE INFO

MSC:
39A06

Keywords:
Discrete system
Z-transform
Multiple delays
Matrix polynomial

ABSTRACT

In the present paper, a system of nonhomogeneous linear difference equations with any finite number of constant delays and linear parts given by pairwise permutable matrices is considered. Representation of its solution is derived in a form of a matrix polynomial using the $\mathcal{Z}$-transform. So the recent results for one and two delays, and an inductive formula for multiple delays are unified. The representation is suitable for theoretical as well as practical computations.

1. Introduction

Recently, Khusainov and Diblík [10] derived a solution of a difference equation with a delay using a matrix polynomial. This representation lead to results on controllability, observability, stability etc. (see e.g. [5,12–14]). Motivated by these applications, results on representation of a solution of a difference equation with more than one delay appeared. However, these contain an explicit formula for two delays [6] or an inductively built formula for the case of $n \in \mathbb{N}$ delays [11]. Although the latter result was successfully applied in the same paper to prove results on exponential stability, for practical computations it seems to be not very suitable. The aim of this paper is to provide a representation of a solution of a nonhomogeneous linear difference equation with multiple delays in a closed form, and so to generalize results of [6,10]. The results of this paper may also help to explore the stability properties of delayed difference equations [3,4].

To achieve our goal we use the $\mathcal{Z}$-transform and its inverse. By this way, we also avoid investigating particular cases of the time k resulting in vast proofs as in [6]. We note that as in [6,11] we suppose that the linear parts are given by pairwise permutable matrices. This allows to change the order when multiplying matrices.

Throughout the paper we denote Θ and $\mathbb{I}$ the $N \times N$ zero and identity matrix, respectively, $\mathbb{Z}_a^b := \{a, a+1, \dots, b\}$ for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a \le b$, and $\mathbb{Z}_a^b = \emptyset$ if $a > b$. We suppose the properties of an empty sum $\sum_{i=a}^{b} z(i) = 0$ and an empty product $\prod_{i=a}^{b} z(i) = 1$ for integers $a > b$, where $z(i)$ is a given function which does not have to be defined for each $i \in \mathbb{Z}_b^a$ in this case. Standardly, $\Delta x(k) := x(k+1) - x(k)$ is the forward difference operator, and $\lfloor \cdot \rfloor$ denotes the floor function. We shall denote $\|v\| = \max_{i \in \mathbb{Z}_1^N} |v^i|$ the maximum norm of a vector $v = (v^1, \dots, v^N) \in \mathbb{R}^N$, and $\|B\|$ an induced matrix norm of an $N \times N$ matrix B.

Let us recall the above-mentioned results for one, two and multiple delays.

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http://dx.doi.org/10.1016/j.amc.2016.09.019
0096-3003/

Theorem 1.1 (see [10]). *Let $m \geq 1$, B be a constant $N \times N$ matrix, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ and $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^N$ be given functions. Solution $x(k)$ of the Cauchy problem consisting of the equation*

$$\Delta x(k) = Bx(k-m) + f(k), \qquad k \geq 0 \tag{1.1}$$

and the initial condition

$$x(k) = \varphi(k), \qquad k \in \mathbb{Z}_{-m}^0 \tag{1.2}$$

has the form

$$x(k) = \mathrm{e}_m^{Bk}\varphi(-m) + \sum_{j=-m+1}^{0} \mathrm{e}_m^{B(k-m-j)}\Delta\varphi(j-1) + \sum_{j=1}^{k} \mathrm{e}_m^{B(k-m-j)} f(j-1)$$

for $k \in \mathbb{Z}_{-m}^\infty$ where e_m^{Bk} is the discrete delayed matrix exponential given by

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta, & \text{if } k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ \mathbb{I} + \sum_{j=1}^{l} B^j \binom{k-(j-1)m}{j}, & \text{if } k \in \mathbb{Z}_{(l-1)(m+1)+1}^{l(m+1)},\ l \in \mathbb{Z}_0^\infty. \end{cases} \tag{1.3}$$

It was proved in [10] that e_m^{Bk} is a matrix solution of the equation

$$\Delta X(k) = BX(k-m), \qquad k \in \mathbb{Z}_{-m}^\infty, \tag{1.4}$$

satisfying the initial condition

$$X(k) = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ \mathbb{I}, & k \in \mathbb{Z}_{-m}^0. \end{cases} \tag{1.5}$$

Theorem 1.2 (see [6]). *Let $m_2 > m_1 \geq 1$, B_1 and B_2 be permutable, i.e. $B_1B_2 = B_2B_1$, $f : \mathbb{Z}_0^\infty \to \mathbb{R}^N$ and $\varphi : \mathbb{Z}_{-m_2}^0 \to \mathbb{R}^N$ be given functions. Solution $x(k)$ of the Cauchy problem*

$$\begin{aligned} \Delta x(k) &= B_1x(k-m_1) + B_2x(k-m_2) + f(k), \qquad k \geq 0 \\ x(k) &= \varphi(k), \qquad k \in \mathbb{Z}_{-m_2}^0 \end{aligned}$$

has the form

$$x(k) = \sum_{j=0}^{m_2} \tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(k+j)} w_j + \sum_{j=1}^{k} \tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(k-j)} f(j-1), \qquad k \in \mathbb{Z}_{-m_2}^\infty$$

where

$$\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2k} = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-1}, \\ \mathbb{I}, & k \in \mathbb{Z}_0^{m_1}, \\ \mathbb{I} + \sum\limits_{i=0}^{\lfloor \frac{k-1}{m_1+1} \rfloor} \sum\limits_{j=0}^{\lfloor \frac{k-1}{m_2+1} \rfloor} B_1^i B_2^j & \\ \quad \times \binom{i+j}{i} \left[B_1 \binom{k-m_1(i+1)-m_2j}{i+j+1} + B_2 \binom{k-m_1i-m_2(j+1)}{i+j+1} \right], & k \in \mathbb{Z}_{m_1+1}^\infty \end{cases}$$

and

$$w_k = \begin{cases} \Delta\varphi(-k-1) - \Delta\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(-k+m_2-1)}\varphi(-m_2) & \\ - \sum\limits_{j=-m_2}^{-k-m_1-2} \Delta\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2(-k-j-2)}\Delta\varphi(j), & k \in \mathbb{Z}_0^{m_2-m_1-1} \\ \Delta\varphi(-k-1), & k \in \mathbb{Z}_{m_2-m_1}^{m_2-1}, \\ \varphi(-k), & k = m_2. \end{cases}$$

From [6] we know that $\tilde{\mathrm{e}}_{m_1m_2}^{B_1B_2k}$ is a matrix solution of the equation

$$\Delta X(k) = B_1X(k-m_1) + B_2X(k-m_2), \qquad k \in \mathbb{Z}_0^\infty$$

and, clearly, it satisfies

$$X(k) = \begin{cases} \Theta, & k \in \mathbb{Z}_{-\infty}^{-1}, \\ \mathbb{I}, & k = 0. \end{cases} \tag{1.6}$$