



Unconditional superconvergence analysis of conforming finite element for nonlinear parabolic equation



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ABSTRACT

Galerkin finite element approximation to nonlinear parabolic equation is studied with a linearized backward Euler scheme. The error between the exact solution and the numerical solution is split into two parts which are called the temporal error and the spatial error through building a time-discrete system. On one hand, the temporal error derived skillfully leads to the regularity of the time-discrete system solution. On the other hand, the τ -independent spatial error and the boundedness of the numerical solution in L^∞ -norm is deduced with the above achievements. At last, the superclose result of order $O(h^2 + \tau)$ in H^1 -norm is obtained without any restriction of τ in a routine way. Here, h is the subdivision parameter, and τ , the time step.

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1. Introduction

Consider the following nonlinear parabolic equation:

$$\begin{cases} u_t - \nabla \cdot (a(u) \nabla u) = f(u), & (X, t) \in \Omega \times (0, T], \\ u = 0, & (X, t) \in \partial\Omega \times (0, T], \\ u(X, 0) = u_0(X), & X \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with the boundary $\partial\Omega$, $0 < T < \infty$ and $X = (x, y)$. $a(u)$, $f(u)$, $u_0(X)$ are known smooth functions. Assume that $a(u)$ is a twice continuously differentiable with respect to u , $0 < a_0 \leq a(u) \leq a_1$ for certain positive constants a_0 , a_1 . In addition, $f(u)$ is locally Lipschitz continuous in u .

For nonlinear parabolic problems, the linearized Galerkin finite element method (FEM) is a preference. For example, Thomée [1] constructed linearized scheme for (1.1) and obtained a superclose result of order $O(h^2 + \tau)$ in L^2 -norm for conforming linear triangle element. Rachford [2] discussed several linearized variants of a Crank–Nicolson discrete time Galerkin procedure, where optimal rate estimates were derived. In [3], optimal order error estimates in L^2 -norm and H^1 -norm were obtained by using a linearized scheme with a nonlinear H^1 projection of the exact solution for conforming elements. However, in studying the error estimate of a linearized scheme, the boundedness of numerical solution in L^∞ -norm or a stronger norm is often required. If a priori estimate for numerical solution in such a norm can't be obtained, one may employ the inverse inequality to deal with this issue, which will result in some time-step restrictions. For instance, Rachford [2] and Luskin [3] studied the case $\tau = O(h^{q/2})$ ($q \geq 4$) and $\tau^4 = O(h^q)$ ($q \leq 3$), respectively. Furthermore, for many other nonlinear physical systems, the time-dependent optimal error estimates have been extensively investigated, such as

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nonlinear Sobolev problems [4,5], nonlinear Schrödinger equations [6,7] and Navier–Stokes equations [8–10]. In order to avoid such defect, the so-called splitting technique [11] was used to solve a class of nonlinear parabolic equations and obtain the unconditional error estimates, where a corresponding time-discrete system was constructed to split the error into two parts, the temporal error and the spatial error. More precisely, the spatial error is τ -independent and the numerical solution can be bounded in L^∞ -norm by an inverse inequality unconditionally. Later, Sun et al. [11–17] also applied such idea to other equations. However Sun et al. [11–17] only arrived at optimal estimate without referring to the superclose character. Recently, Shi et al. [18] discussed the unconditional superclose property for Sobolev equation with conforming mixed FEM, in which the authors first derived the estimate $\|\tilde{\partial}_t(U_h^i - I_h u^i)\|_1$ and then employed $U_h^n - I_h u^n = \tau \sum_{i=1}^n \tilde{\partial}_t(U_h^i - I_h u^i)$ to get the final results (the notations appeared herein will be defined below). But this method is difficult to be extended to parabolic equation for it relies on the nature of the equation too much.

Different from Thomée [1] and Shi et al. [18], the main aim of this paper is to study the unconditional superconvergence analysis of u in H^1 -norm for (1.1) with bilinear conforming FE. Motivated by the idea of splitting technique in [11–17], we introduce a time discrete system with solution of variable U^n to split the error $u^n - U_h^n$ into the temporal error and the spatial error. Firstly, we estimate the temporal error which leads to the regularity of U^n . Secondly, we get the spatial error, $\tau \sum_{i=1}^n \|\tilde{\partial}_t U_h^{n-1} - \partial_t I_h U^{n-1}\|_0^2 = O(h^2)$, which plays a key role in getting rid of the constraint of τ . At last, we use different strategy to reach the order of $O(h^2 + \tau)$. It is also worthy to be emphasized that, the requirement of a bounded smooth domain Ω in [11–17] is much stringent than the condition of this work.

The paper is organized as follows. In Section 2, a brief description of conforming bilinear FE and an indispensable lemma are given. At the same time, a linearized Euler Galerkin FE scheme is established. In Section 3, a time-discrete system is developed and the corresponding temporal errors are obtained. In Section 4, the unconditional spatial error with order $O(h)$ and superclose result of u with order $O(h^2 + \tau)$ are derived, respectively. In Section 5, some numerical results are provided to show the validity of the theoretical analysis.

2. Conforming FE approximation scheme

Let Ω be a rectangle in (x, y) plane with edges parallel to the coordinate axes, Γ_h be a quasiuniform partition of Ω into rectangular π_h . Denote $h = \max_{\pi_h \in \Gamma_h} \text{diam} \pi_h$ the mesh size. Let V_h be the usual bilinear FE space, $V_{h0} = \{v_h \in V_h, v_h|_{\partial\Omega} = 0\}$ and $I_h: H^2(\Omega) \rightarrow V_h$ be the associated interpolation operator on V_h . Then we have the following lemma which plays an important role in achieving the superconvergence later and it can be found in [19].

Lemma 1. If $u \in H^3(\Omega)$, then for all $v_h \in V_{h0}$, there holds

$$(\nabla(u - I_h u), \nabla v_h) = O(h^2) \|u\|_3 \|\nabla v_h\|. \quad (2.1)$$

Let $\{t_n : t_n = n\tau; 0 \leq n \leq N\}$ be a uniform partition of $[0, T]$ with the time step $\tau = T/N$. We denote $\sigma^n = \sigma(X, t_n)$. For a sequence of functions $\{\sigma^n\}_{n=0}^N$, we remark $\tilde{\partial}_t \sigma^n = \frac{\sigma^n - \sigma^{n-1}}{\tau}$, $n = 1, 2, \dots, N$. With these notations, we develop the linearized Galerkin FEM to problem (1.1): seek $U_h^n \in V_{h0}$, such that

$$(\tilde{\partial}_t U_h^n, v_h) + (a(U_h^{n-1}) \nabla U_h^n, \nabla v_h) = (f(U_h^{n-1}), v_h), n = 1, 2, \dots, N, \forall v_h \in V_{h0}, \quad (2.2)$$

where $U_h^0 = I_h u_0$. Obviously, a linear system with certain constant coefficients needs to be solved now.

3. Error estimates for time-discrete system

In this section, we introduce the following time-discrete system:

$$\begin{cases} \tilde{\partial}_t U^n - \nabla \cdot (a(U^{n-1}) \nabla U^n) = f(U^{n-1}), & X \in \Omega, n \geq 1, \\ U^n = 0, & X \in \partial\Omega, n \geq 1, \\ U^0(X) = u_0(X), & X \in \Omega. \end{cases} \quad (3.1)$$

The above system can be viewed as a system of linear elliptic equations, and the existence and uniqueness of solution can be proved immediately. In what follows, we set $e^n = u^n - U^n$ ($n = 0, 1, 2, \dots, N$), analyze $\|u^n - U^n\|_i$ ($i = 0, 1, 2$) and give the regularity result of U^n .

Theorem 1. Let u and U^m ($m = 0, 1, 2, \dots, N$) be the solutions of (1.1) and (3.1), respectively, $u \in L^2(0, T; H^3(\Omega))$, $u_t \in L^\infty(0, T; H^2(\Omega))$, $u_{tt} \in L^2(0, T; L^2(\Omega))$, then for $m = 1, \dots, N$, there exists τ_0 such that when $\tau < \tau_0$, we have

$$\frac{1}{\sqrt{\tau}} \|\nabla e^1\|_0 + \|e^1\|_2 + \|\nabla e^m\|_0 + \left(\tau \sum_{i=2}^m \|e^i\|_2^2 \right)^{\frac{1}{2}} + \tau^{\frac{1}{2}} \|e^m\|_2 \leq C_0 \tau, \quad (3.2)$$

and

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