Applied Mathematics and Computation 294 (2017) 299–309

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Further results regarding the sum of domination number and average eccentricity

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ARTICLE INFO

MSC:
05C12
05C35
92E10

Keywords:
Distances
Average eccentricity
Domination number
AGX conjectures

ABSTRACT

The average eccentricity of a graph G, denoted by $ecc(G)$, is the mean value of eccentricities of all vertices of G. Let $D_{n,\, i}$ be the n-vertex tree obtained from a path $P_{n-1} = v_1 v_2 \dots v_{n-1}$ by attaching a pendent vertex to v_i. In [13], it was shown that the maximum value for the sum of domination number and average eccentricity among n-vertex (connected) graphs is attained by $D_{n,\, 3}$ when $n \equiv 0 \pmod 3$, and attained by the path P_n when $n \not\equiv 0 \pmod 3$. In this paper, we will further determine the second maximum value for the sum of domination number and average eccentricity among n-vertex (connected) graphs. It is interesting that the graphs attaining that second maximum value have three cases, which is $D_{n,\, 6}$ when $n \equiv 0 \pmod 3$, $D_{n,\, 3}$ when $n \equiv 1 \pmod 3$, and T_n when $n \equiv 2 \pmod 3$, where T_n is the n-vertex tree obtained from a path $P_{n-2} = v_1 v_2 \dots v_{n-2}$ by attaching a pendent vertex to v_3, and a pendent vertex to v_{n-4}.

1. Introduction

Let G be a simple connected graph, whose vertex set is denoted by $V(G)$. For vertices $u, v \in V(G)$, let $d_G(u, v)$ be the distance from u to v in G, i.e., the length of a shortest path between u and v in G. The eccentricity of a vertex v in G, denoted by $\varepsilon_G(v)$, is the maximum distance from v to any other vertex in G, i.e.,

$$\varepsilon_G(v) = \max_{u \in V(G)} d_G(u, v)\,.$$

Historically, graph theory has a close relationship with other subjects, such as physics, chemistry, genetics, etc. As an application of graph theory in theoretical chemistry, one usually uses a graph to represent the molecule graph of a chemical compound. The topological indices are numbers associated with chemical structures via their hydrogen-depleted graphs, which have been often used in the modeling of various structure-property relationships of chemical compounds, e.g., the physico-chemical, biological, toxicologic properties, etc.

Among a variety of topological indices, some are based on the degrees of vertices, e.g., Zagreb indices [6,14,19], Randić index [27,29], etc., and some are based on the distances of vertices, e.g., Wiener index [20,33], eccentric connectivity index [28,36], etc.

The Wiener index is recognized as the first proposed topological index [20,33], which is defined as

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v)\,.$$

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http://dx.doi.org/10.1016/j.amc.2016.09.014
0096-3003/

The research on the Wiener index of graphs has a long history, a summary on Wiener index can be found in [10,11], and some recent results on the Wiener-type index of graphs can be referred to [1,24,25,30].

Clearly, the eccentricity of vertices is also a distance-based graph invariant. Now there are several types of topological indices related to the eccentricity of vertices, one of them is the eccentric connectivity index [28], which is defined as

$$\xi^c(G) = \sum_{v \in V(G)} d_G(v)\varepsilon_G(v)\,,$$

where $d_G(v)$ denotes the degree of v in G. These years, the research on the eccentric connectivity index of graphs is rather active.

Zhou and Du [36] established various lower and upper bounds for the eccentric connectivity index of graphs in terms of such graph invariants as the number of vertices, number of edges, degree distance and the first Zagreb index, and they also determined the trees with extremal eccentric connectivity indices. Morgan et al. [26] presented a tight lower bound for the eccentric connectivity index of graphs in terms of the number of vertices and diameter. More results on the eccentric connectivity index of graphs can be referred to [8,21,23], and some recent results on the eccentricity-type index of graphs can be referred to [17,35].

The average eccentricity of a graph G on n vertices, denoted by $ecc(G)$, is the mean value of eccentricities of all vertices of G, i.e.,

$$ecc(G) = \frac{1}{n} \sum_{v \in V(G)} \varepsilon_G(v)\,.$$

As another topological index based on the eccentricity of vertices, the average eccentricity of graphs also receives a lot of attentions.

Ilić [22] introduced two graph transformations that increase/decrease the average eccentricity of graphs, which contribute to the analysis of extremal properties of the average eccentricity of graphs. Recently, Dankelmann and Mukwembi [9] presented several upper bounds on the average eccentricity of graphs in terms of the independence number, chromatic number, domination number and connected domination number, respectively.

The trees and unicyclic graphs are the simplest connected graphs, they are always the focus of research in graph theory. So it is natural that the average eccentricities of trees and unicyclic graphs are of particular interest. Tang and Zhou [31] presented some lower and upper bounds for the average eccentricity of trees when the diameter, number of pendent vertices and matching number are, respectively, fixed, and determined the n-vertex trees with the first four smallest and the first $n/2$th largest average eccentricities, and later they [32] determined the n-vertex unicyclic graphs with the first $(\lfloor \frac{n}{2} \rfloor - 1)$th largest average eccentricities. More results on the average eccentricity of graphs can be referred to [7,18,34].

AutoGraphiX (AGX) [2,3] is an interactive software developed by GERAD group from Montréal. It mainly uses the variable neighborhood search metaheuristic and data analysis methods to find extremal graphs with respect to one or more graph invariants, based on that, one may get some potential results and propose conjectures in graph theory.

A number of conjectures proposed by AGX, which is called AGX conjectures, were presented in [2]. Recently there are vast researches regarding AGX conjectures and a series of papers on various graph invariants, e.g., independence number [4], the spectrum of graphs [5], Randić index [15].

In particular, there are a lot of conjectures in [2] with respect to the average eccentricity of graphs. Ilić [22] resolved four AGX conjectures about the average eccentricity and other graph parameters, including the clique number, independence number and Randić index, and refuted one AGX conjecture about the average eccentricity and the minimum vertex degree. Later, as a sequel of [22], Du and Ilić [12] resolved another five AGX conjectures about the average eccentricity and other graph parameters, including the independence number, chromatic number and Randić index, and refuted two AGX conjectures about the average eccentricity and the spectral radius.

The closed neighborhood of a vertex v in a graph G contains v and all of its neighbors in G. A dominating set of a graph is a vertex subset whose closed neighborhood contains all vertices of the graph. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [16].

It is also worth mentioning that Aouchiche [2] proposed a conjecture about the sum of domination number and average eccentricity (which is Conjecture A.482 in [2]), and recently Ilić [22] corrected Aouchiche's conjecture, and proposed a revised conjecture as shown in Conjecture 1.1. Later, Du and Ilić [13] showed that the revised conjecture is true, more precisely, they determined the maximum value for the sum of domination number and average eccentricity.

Let P_n be the path on n vertices. Let $D_{n,\,i}$ be the n-vertex tree obtained from the path $P_{n-1} = v_1 v_2 \dots v_{n-1}$ by attaching a pendent vertex to v_i, where $2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. Let T_n be the n-vertex tree obtained from the path $P_{n-2} = v_1 v_2 \dots v_{n-2}$ by attaching a pendent vertex to v_3, and a pendent vertex to v_{n-4}, where $n \geq 8$.

Conjecture 1.1 [22]**.** *Let G be an n-vertex connected graph, where $n \geq 4$. Then*

$$\gamma(G) + ecc(G) \leqslant \begin{cases} \lceil \frac{n}{3} \rceil + \frac{1}{n} \lfloor \frac{3}{4} n^2 - \frac{1}{2} n \rfloor & \text{if } n \not\equiv 0 \pmod 3 \\ \frac{n}{3} + 2 - \frac{3}{n} + \frac{1}{n} \lfloor \frac{3}{4}(n-1)^2 - \frac{1}{2}(n-1) \rfloor & \text{if } n \equiv 0 \pmod 3 \end{cases}$$

with equality if and only if $G \cong P_n$ when $n \not\equiv 0 \pmod 3$ and $G \cong D_{n,\,3}$ when $n \equiv 0 \pmod 3$.