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Optimal asymptotic Lebesgue constant of Berrut's rational interpolation operator for equidistant nodes

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ABSTRACT

In approximation theory, the Lebesgue constant of an interpolation operator plays an important role. The Lebesgue constant of Berrut's interpolation operator has been extensive studied. In the present work, by introducing a new method, we obtain an optimal asymptotic Lebesgue constant of Berrut's rational interpolant at equidistant nodes.

1. Introduction

Berrut's rational interpolant [1–4] to approximate a function $f:[a,b]\to\mathbb{R}$ at the $n+1$ distinct interpolation nodes

$$a = x_0 < x_1 < \cdots < x_n = b$$

is defined by

$$r_n(f,x) = \sum_{i=0}^{n} f(x_i) b_i(x),$$

where

$$b_i(x) = \frac{(-1)^i}{x-x_i} \Big/ \sum_{j=0}^{n} \frac{(-1)^j}{x-x_j}.$$

The Lebesgue constant [5–7,10] of this interpolation operator is

$$\Lambda_n = \max_{a\le x\le b} \sum_{i=0}^{n} |b_i(x)|.$$

In approximation theory, the Lebesgue constant of interpolation operators plays an important role. Corresponding results abound in the literature [8,9,11–15]. The more interesting bound is the upper one, since, when small, it guarantees the well-conditioning of the interpolation process. For equidistant nodes, Bos et al. [5] have obtained the following upper and lower bounds of the Lebesgue constant:

$$\frac{2n}{4+n\pi}\ln(n+1) \le \Lambda_n \le 2 + \ln(n). \tag{1}$$

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In the recent work [16], the following tighter upper bound has been obtained:

$$\Lambda_n \le \frac{1}{1+\frac{\pi^2}{24}} \ln(n+1) + 1, \quad n \ge 174. \tag{2}$$

However the factor $\frac{1}{1+\frac{\pi^2}{24}}$ in the leading term $\ln(n+1)$ is not optimal. Based on some numerical experiments [5,16], a few researchers have guessed that the optimal factor could be $\frac{2}{\pi}$. In this paper, by introducing a new idea, we prove this conjecture. The main result is as follows:

$$\Lambda_n \le \frac{2}{\pi} \ln(n+2) + 2.9468, \quad n \ge 46. \tag{3}$$

Combining (3) with (1), one can readily see that

$$\lim_{n\to\infty} \frac{\Lambda_n}{\ln n} = \frac{2}{\pi}.$$

Thus, the factor $\frac{2}{\pi}$ in (3) can not be replaced by a smaller one when n is sufficiently large. This shows the validity of the conjecture. The constant term 2.9468 in the inequality (3) is not optimal. It seems a more difficult problem to find the optimal one. Finally, we should point out a recent result given by Ibrahimoglu and Cuyt [17]. They obtained a "precise growth formula"

$$\frac{2(\ln(n+1)+\ln 2+\gamma)}{\pi+\frac{4}{n+3}} \le \Lambda_n \simeq \frac{2\left(\ln(n+1)+\ln 2+\gamma+\frac{1}{24n}\right)}{\pi-\frac{4}{n+2}},$$

where $\gamma = 0.5772\ldots$ is the Euler–Mascheroni constant. However, their result does not stand on firm ground since their work depends on computer observations; they did prove an upper bound only for the Lebesgue function in the center of every interval between two nodes, not for Λ_n.

2. The result and its proof

2.1. Some elementary inequalities

To obtain the result (3), we need some elementary expressions. Let

$$\alpha_n(x) := \ln(n - \frac{1}{2} - x) + \ln(x + \frac{3}{2}). \tag{4}$$

$\alpha_n(x)$ is an increasing function on the interval $x \in [1, \frac{n}{2} - 1]$ $(n \ge 10)$, which yields for an integer k

$$2 < \alpha_n(k) \le 2\ln(n+1) - 2\ln 2, \qquad 1 \le k \le \frac{n}{2} - 1. \tag{5}$$

For simplicity, we introduce the function

$$s_n(t) := \sum_{i=1}^{n} \left(\frac{(-1)^i}{t-2i} + \frac{(-1)^i}{t+2i} \right), \tag{6}$$

which can be modified to

$$s_n(t) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2i} \left(\frac{1}{1-\frac{t}{2i}} - \frac{1}{1+\frac{t}{2i}} \right) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} \left(\frac{t}{2i} + \left(\frac{t}{2i}\right)^3 + \cdots \right) = \sum_{m=1}^{\infty} a_m t^{2m-1},$$

where $a_m := \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i^{2m}} \left(\frac{1}{2}\right)^{2m-1} > 0$. It is easy to see from the above expression that

$$s_n^{(k)}(t) \ge 0, \quad k = 0, 1, 2, \ldots. \quad t \in \mathbb{R}. \tag{7}$$

Thus, $s_n^{(k)}(t)$ is an increasing function on $t \in [0, 1]$ for a fixed k. From (6), it is not difficult to obtain

$$s_n\left(\frac{2}{3}\right) = \left(\frac{3}{4} - \frac{3}{8} - \frac{3}{10} + \frac{3}{14}\right) + \left(\frac{3}{16} - \frac{3}{20} - \frac{3}{22} + \frac{3}{26}\right) + \cdots > \frac{2}{7}, \tag{8}$$

$$s_n'\left(\frac{2}{3}\right) > \left(\frac{9}{16} + \frac{9}{64} - \frac{9}{100} - \frac{9}{196}\right) + \left(\frac{9}{256} + \frac{9}{400} - \frac{9}{484} - \frac{9}{676}\right) + \cdots > \frac{9}{16}, \tag{9}$$

and

$$s_n'(1) = 1 + (-1)^{n+1} \frac{1}{(2n+1)^2} \le 1 + \frac{1}{(2n+1)^2}. \tag{10}$$