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Amplitude–frequency relationship obtained using Hamiltonian approach for oscillators with sum of non-integer order nonlinearities

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ABSTRACT

In this paper the Hamiltonian approximate analytical approach is extended for solving vibrations of conservative oscillators with the sum of integer and/or non-integer order strong nonlinearities. Solution of the nonlinear differential equation is assumed in the form of a trigonometric function with unknown frequency. The frequency equation is obtained based on the hypothesis that the derivative of the Hamiltonian in amplitude of vibration is zero. The accuracy of the approximate solution is treated with two different approaches: comparing the analytical value for period of vibration with the period obtained numerically and developing an error estimation method based on the ratio between the averaged residual function and the total constant energy of system. The procedures given in the paper are applied for two types of examples: an oscillator with a strong nonlinear term and an oscillator where the nonlinearity is of polynomial type.

1. Introduction

Recently, a significant number of methods developed for weakly nonlinear oscillators (see [1]) are adopted for solving strong nonlinear oscillators, such as: Lindstedt–Poincare method [2], harmonic balance method ([3,4]), homotopy perturbation method [5,6], variational iteration method ([7]), Hamiltonian approach ([8,9]), Krylov–Bogolubov method ([10,11]). Mickens [12] presented their application on the oscillator with the nonlinearity of certain non-integer order. Further investigation were directed toward generalization of the model where the nonlinearity has the order, which is any rational number (integer or non-integer) ([13–19]).

In this paper an oscillator whose nonlinearity is of polynomial type, with orders which are rational numbers, is considered. The model of the oscillator is

$$\ddot{x} + f(x) = 0, \tag{1}$$

with initial conditions

$$x(0) = A, \dot{x}(0) = 0, \tag{2}$$

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where $f(x)$ is the conservative force given by (see [19])

$$f(x) = \sum_{k=1}^{p} c_k^2 x|x|^{\alpha_k - 1}, \tag{3}$$

and c_k^2 are positive small or large constants and the order of nonlinearities α_k are rational numbers (integers or non-integers). To solve the Eq. (1) the Hamiltonian approach developed by He [8] is extended for obtaining an amplitude–frequency relationship for oscillators with the sum of integer and rational positive power terms. It is necessary to predict the accuracy of the analytically obtained approximate solution. It is not an easy task, when the exact closed form solution of the problem is not known. For that case, usually, the approximate solution is compared with a numerically obtained one. Using the Runge–Kutta procedure the numerical solution of (1) for certain initial conditions (3) is calculated. The obtained numerical solution and the approximate solutions are compared and their difference is located in certain values of time. It would be of interest to predict the error of the approximate solution without knowledge of the exact solution of the oscillator, independently of the time. The average error between the approximate and the exact solution has to be determined. There are many error estimating procedures based on the value of the mean square residual [20,21]. Atanackovic and Achenbach [21] proposed an error estimation method for some vibration problems. Recently, Farzaneh and Tootoonchi [22] introduced the method for minimization of the error for the frequency of the Duffing oscillator. Thereafter, Akbarzade and Langari [23] and Yazdi and Tehrani [24] successfully applied the technique to several nonlinear problems.

In this paper a comprehensive method for error estimation, which does not depend on the exact solution, is introduced. The method is based on averaging of the residual function of the conservative oscillator. The error is estimated as the integral of the square of the difference between the calculated and total energy of the oscillator, i.e., corresponding Hamiltonians, over the period of vibration. The error is a function of initial amplitude of vibration, coefficients and orders of nonlinearity. The relative error is the rate between the square root of the calculated integral and the initial energy of the oscillator. In the paper two examples are treated: an oscillator with a strong nonlinear term and an oscillator where the nonlinearity is of polynomial type. The accuracy of the approximate solution is tested applying the suggested procedure.

2. The approximate solution

Eqs. (1) with (3) describes a conservative oscillator with kinetic energy

$$K = \frac{\dot{x}^2}{2}, \tag{4}$$

and potential energy

$$F(x) = \sum_{k=1}^{p} \frac{c_k^2}{\alpha k + 1}|x|^{\alpha k+1}. \tag{5}$$

The total energy of the oscillator corresponds to its Hamiltonian

$$H = \frac{1}{2}\dot{x}^2 + F(x) = \frac{1}{2}\dot{x}^2 + \sum_{k=1}^{p} \frac{c_k^2}{\alpha_k + 1}|x|^{\alpha_k+1}. \tag{6}$$

For the conservative oscillator the total energy keeps unchanged during the oscillation. It implies that the Hamiltonian is constant, i.e., $H = H_0 = const.$, and

$$\frac{1}{2}\dot{x}^2 + \sum_{k=1}^{p} \frac{c_k^2}{\alpha_k + 1}|x|^{\alpha_k+1} - H_0 = 0, \tag{7}$$

where due to initial conditions (2)

$$H_0 = \sum_{k=1}^{p} \frac{c_k^2}{\alpha_k + 1}A^{\alpha_k+1}. \tag{8}$$

Very often it is impossible to obtain the closed form analytic solution of (7). Trial solution is assumed in the form

$$x(t) = A\cos(\omega t), \tag{9}$$

where ω is the unknown frequency of vibration and A is the amplitude of the oscillator. Substituting (9) into (6), we have

$$\tilde{H} = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + \sum_{k=1}^{p} \frac{c_k^2}{\alpha_k + 1}A^{\alpha_k+1}\left[\sqrt{\cos^2(\omega t)}\right]^{\alpha_k+1}. \tag{10}$$

Usually, the frequency ω is determined from the derivative of Eq. (7)

$$\frac{\partial \tilde{H}}{\partial A} = 0. \tag{11}$$