Applied Mathematics and Computation 293 (2017) 57–64

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Construction of positivity preserving numerical method for stochastic age-dependent population equations

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ARTICLE INFO

MSC:
60H35
60H30
60H10
65C30

Keywords:
Stochastic age-dependent population equations
Numerical solution
Positivity preserving
Convergence

ABSTRACT

The aim of this paper is to construct a numerical method preserving positivity for stochastic age-dependent population equations. We use the balanced implicit numerical techniques to maintain the nonnegative path of the exact solution. It is proved that the Balanced Implicit Method (BIM) preserves positivity and converges with strong order 1/2 under given conditions. Finally, two examples are simulated to verify the positivity and efficiency of the proposed method.

1. Introduction

Positivity is a fundamental characteristic in many real world system. For example, the biomass in the population modeling is inherently non-negative, and the stock option models usually require positivity due to financial interpretation.

In recent years, a number of numerical methods that aim to preserve the nonnegative path of the exact solution of stochastic differential equations (SDEs) have been developed. For example, Schurz [15] constructed the nonnegative numerical solutions for SDEs through balanced method, and he [16] also investigated that the BIM preserves the qualitative properties of underlying stochastic logistic equations. Milstein et al. [11] proved that the BIM can preserve positivity for stiff SDEs. Kahl et al. [10] showed that a class of Milstein methods can preserve positivity for some interest rate models in finance. Higham [4] proposed a new Milstein type scheme for simulating a class of non-negativity financial models. According to Halidias' recent researches [5,6], a novel semi-discrete method preserving positivity for some SDEs is developed. For more information about positivity preservation we refer the readers to [1–3,7,9,14].

In this paper, we will consider the following stochastic age-dependent population equation.

$$\frac{\partial P}{\partial t}+\frac{\partial P}{\partial a}=-\mu(t,a)P+f(t,P)+g(t,P)\frac{dW_t}{dt} \tag{1.1}$$

where $P(t, a)$ denotes the population density of age a at t, $\mu(t, a)$ denotes the mortality rate of age a at t, $f(t, P)$ denotes the effects of external environment for population system, and $g(t, P)$ is a diffusion coefficient.

As far as this system (1.1) is considered, we have found many numerical results by far. For instance, Li et al. [8] investigated the Euler method to stochastic age-dependent population equations. Pang et al. [12] studied the semi-implicit

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http://dx.doi.org/10.1016/j.amc.2016.08.010

Euler method for stochastic age-dependent population equations. Rathinasamy et al. [13] analyzed split-step θ method for stochastic age-dependent population equations with Markovian switching. Zhang et al. [18] developed split-step backward Euler (SSBE) method for stochastic age-dependent capital system with Markovian switching.

However, in the fore-mentioned numerical methods of system (1.1), such appearances as negative paths or unbounded solutions are sometimes observed in contrast to the qualitative behaviour of the exact solution. Therefore, preservation of positivity is a desirable property for numerical approximations.

To the best of our knowledge, there are no results about numerical methods preserving positivity for stochastic age-dependent population equations. Thus, we will construct a balanced implicit method preserving positivity for system (1.1).

The outline of the paper is as follows. In Section 2, we will introduce some preliminary results which are essential for our analysis. Section 3 will give the definition of the life time of numerical solutions. In Section 4, a numerical method preserving positivity for stochastic age-dependent population equation will be constructed. And next Section, the strong convergence and its order of the balanced method are proved. Finally, two numerical examples will be given to verify the positivity and efficiency of the balanced method.

2. Preliminaries

First, we introduce the following notation. Let

$$V = H^1([0,A]) \equiv \left\{ \varphi | \varphi \in L^2([0,A]),\ \frac{\partial \varphi}{\partial x_i} \in L^2([0,A]) \right.$$
$$\left. \text{where } \frac{\partial \varphi}{\partial x_i} \text{ is generalized partial derivatives} \right\},$$

V is a Sobolev space. $H = L^2[0,A]$ such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

V' is the dual space of V. We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norm in V, H and V' respectively; by $\langle \cdot, \cdot \rangle$ the duality product between V, V', and by $(\cdot, \cdot)$ the scalar product in H. For an operator $B \in \mathcal{L}(M, H)$ be the space of all bounded linear operators from M into H, we denote by $\|B\|_2$ the Hilbert–Schmidt norm, i.e.,
$\|B\|_2 = tr(BWB^T)$.

Now we define $P_t = \lim_{s\to t} P(s,a)$, $\frac{\partial P_t}{\partial a} = \lim_{s\to t} \frac{\partial P_s}{\partial a}$ and then consider the following stochastic age-dependent population equation

$$\begin{cases} \mathrm{d}_t P = \left[-\frac{\partial P_t}{\partial a} - \mu(t,a) P_t + f(t, P_t) \right] \mathrm{d}t + g(t, P_t) \mathrm{d}W_t, & (t,a) \in Q, \\ P(0,a) = P_0(a), & a \in [0,A], \\ P(t,0) = \int_0^A \beta(t,a) P(t,a) \mathrm{d}a, & \mathrm{t} \in [0, \mathrm{T}], \end{cases} \tag{2.1}$$

where $\mathrm{d}_t P = \frac{\partial P}{\partial t}$, $T > 0$, $A > 0$, $Q = (0,T) \times (0,A)$, $\beta(t, a)$ denotes the fertility rate of age a at t.

For the existence and uniqueness of the solution, we always assume that the following conditions are satisfied:

(i) $f(t,0) = 0, g(t,0) = 0$;

(ii) (The Lipschitz condition) There is a constant $K > 0$, for all $x, y \in H$, such that

$$|f(t,x) - f(t,y)| \vee \|g(t,x) - g(t,y)\|_2 \le K|x-y|; \tag{2.2}$$

(iii) $\mu(t, a)$, $\beta(t, a)$ are continuous in $\overline{G}$ (the closure of G) such that

$$0 \le \mu_0 \le \mu(t,a) \le \bar{\alpha} < \infty, 0 \le \beta(t,a) \le \bar{\beta} < \infty; \tag{2.3}$$

(iv) (Coercivity condition) There is a constant $\alpha > 0$, $\xi > 0$, $\lambda \in \mathbb{R}$ and a nonnegative continuous function $\gamma(t)$, $t \in \mathbb{R}^+$, $\forall v \in H$ such that

$$2\langle f(t,v), v \rangle \vee \| g(t,v) \|_2^2 \le -\alpha |v|^2 + \lambda |v|^2 + \gamma(t) e^{-\xi t}, \tag{2.4}$$

where for arbitrary $\delta > 0$, $\gamma(t)$ satisfies $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$.

Theorem 2.1. *Under the assumptions (i)-(iv), Eq. (2.1) has a unique positive solution on* $[0, T]$.

Proof. The proof of this theorem is similar to that in [17]. □

3. Life time of numerical solutions

The concept of life time of numerical scheme was formalized by Schurz in [15]. He introduced the notion of an algorithm having eternal lifetime, where we generalized this concept to (2.1) as follows.