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Design and multidimensional extension of iterative methods for solving nonlinear problems☆

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A B S T R A C T

In this paper, a three-step iterative method with sixth-order local convergence for approximating the solution of a nonlinear system is presented. From Ostrowski's scheme adding one step of Newton with 'frozen' derivative and by using a divided difference operator we construct an iterative scheme of order six for solving nonlinear systems. The computational efficiency of the new method is compared with some known ones, obtaining good conclusions. Numerical comparisons are made with other existing methods, on standard nonlinear systems and the classical 1D-Bratu problem by transforming it in a nonlinear system by using finite differences. From this numerical examples, we confirm the theoretical results and show the performance of the presented scheme.

1. Introduction

The design of fixed point iterative methods for solving equations and systems of nonlinear equations is an important and challenging task in the field of numerical analysis. In 1990, Moré [18] proposed a collection of nonlinear problems and most of them are phrased in terms of $F(x) = 0$. On the other hand, Grosan and Abraham [10] also discussed the applicability of the system of nonlinear equations in neurophysiology, kinematics syntheses problem, chemical equilibrium problem, combustion problem and economics modeling problem. In addition, the reactor and steering problems are solved in [2,23] by phrasing these problems in the form of $F(x) = 0$. Moreover, Lin et al. [15] also discussed the applicability of the systems of nonlinear equations in transport theory. These and other more examples allow us to affirm that finding the solution ξ of a nonlinear system $F(x) = 0$ is a classical and difficult problem with many applications in science and engineering, wherein $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently Frechet differentiable function in an open convex set D. Since 1980s', many iterative methods have been constructed for solving nonlinear systems, see, for example, [2,6,10,12,17,22,25] and the references therein. The best known method for finding a solution $\xi \in D$ is Newton's scheme,

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}),\ \ k = 0, 1, 2, \ldots,$$

where $F'(x^{(k)})$ is the Jacobian matrix of function F evaluated at the kth iteration.

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The proliferation of iterative methods for solving nonlinear equations has been spectacular in the last years (in [21] we can see a good overview). Some of these methods can be transferred easily to the context of nonlinear systems, keeping the order of convergence, but others, at least apparently, cannot be extended to multidimensional case. In this paper, we use a procedure that allows this extension for many known multi-point iterative methods designed for solving nonlinear equations, making use of tools such as the divided difference operator. Of course, this translation only has interest if the order of convergence is preserved, as it is the case.

Based on Newton's or Newton-like iterations, some higher order methods for computing a solution of nonlinear system $F(x)=0$ have been proposed in the literature. The aim of these new schemes is to accelerate the convergence or to improve the computational efficiency. For example, among other authors, Montazeri et al. [17] and Hueso et al. [12], developed sixth-order iterative methods requiring two evaluations of function F and two of Jacobian F' per iteration, Sharma and Arora [22] designed a sixth-order method which requires three functional and two Jacobian evaluations per iteration. On the other hand, Wang et al. [25] have constructed a seventh-order derivative free iterative method by using the first order divided difference operator $[x, y; F]$ evaluated three times per iteration.

Specifically, the sixth-order scheme by Montazeri et al. [17], that we denote by MSSM, has three-step and its iterative expression is

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - \left[\frac{23}{8}I - 3[F'(x^{(k)})]^{-1}F'(y^{(k)}) + \frac{9}{8}\left([F'(x^{(k)})]^{-1}F'(y^{(k)})\right)^2\right][F'(x^{(k)})]^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left[\frac{5}{2}I - \frac{3}{2}[F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(x^{(k)})]^{-1}F(z^{(k)}),
\end{aligned} \tag{1}$$

where I is the identity matrix of size $n \times n$.

Hueso et al. in [12] developed several iterative schemes of order six; we use some of them in the numerical section for comparing with our proposed scheme on different test problems. In particular, in [12] the authors present the following method that we denote by HMT1

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - \left[-\frac{1}{2}I + \frac{9}{8}[F'(y^{(k)})]^{-1}F'(x^{(k)}) + \frac{3}{8}[F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(x^{(k)})]^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left[-\frac{9}{4}I + \frac{15}{8}[F'(y^{(k)})]^{-1}F'(x^{(k)}) + \frac{11}{8}[F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(y^{(k)})]^{-1}F(z^{(k)})
\end{aligned} \tag{2}$$

and the three-step scheme, denoted by HMT2

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - \left[\frac{5}{8}I + \frac{3}{8}\left([F'(y^{(k)})]^{-1}F'(x^{(k)})\right)^2\right][F'(x^{(k)})]^{-1}F(x^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left[-\frac{9}{4}I + \frac{15}{8}[F'(y^{(k)})]^{-1}F'(x^{(k)}) + \frac{11}{8}[F'(x^{(k)})]^{-1}F'(y^{(k)})\right][F'(y^{(k)})]^{-1}F(z^{(k)}).
\end{aligned} \tag{3}$$

On the other hand, Wang et al. in [25] describe the following derivative-free seventh-order scheme, that we denote by WZQT

$$\begin{aligned}
y^{(k)} &= x^{(k)} - B^{-1}F(x^{(k)}),\\
z^{(k)} &= y^{(k)} - \left[3I - 2B^{-1}[y^{(k)}, x^{(k)}; F]\right]B^{-1}F(y^{(k)}),\\
x^{(k+1)} &= z^{(k)} - \left[\frac{13}{4}I - B^{-1}[z^{(k)}, y^{(k)}; F]\left(\frac{7}{2}I - \frac{5}{4}B^{-1}[z^{(k)}, y^{(k)}; F]\right)\right]B^{-1}F(z^{(k)}),
\end{aligned} \tag{4}$$

where $B = \left[x^{(k)} + F(x^{(k)}), x^{(k)} - F(x^{(k)}); F\right]$ and $[x, y; F] : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the first order divided difference on D.

In order to compare the different methods under the point of view of the computational cost, we recall the computational efficiency index, CI, introduced by the authors in [6], which combine the efficiency index defined by Ostrowski [20] and the number of products-quotients required per iteration. We define this index as $CI = p^{1/(d+op)}$, where p is the order of convergence, d is the number of functional evaluations and op is the number of products-quotients per iteration. Let us remark that for evaluating function F we need n scalar functional evaluations (the coordinate functions of F), whilst for evaluating Jacobian F' it is necessary to evaluate n^2 functions (all the entries of matrix F'). On the other hand, all the iterative methods for solving nonlinear systems require one or more matrix inversion, that is, one or more linear systems must be solved. So, the number of operations needed for solving a linear system plays in this context an important role.

We recall that the number of products and quotients required for solving a linear system by Gaussian elimination is $\frac{1}{3}n^3 + n^2 - \frac{1}{3}n$, where n is the size of the system. In addition, for solving q linear systems, with the same matrix of