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Unification and extension of intersection algorithms in numerical algebraic geometry

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ABSTRACT

The solution set of a system of polynomial equations, called an algebraic set, can be decomposed into finitely many irreducible components. In numerical algebraic geometry, irreducible algebraic sets are represented by witness sets, whereas general algebraic sets allow a numerical irreducible decomposition comprising a collection of witness sets, one for each irreducible component. We denote the solution set of any system of polynomials $f : \mathbb{C}^N \to \mathbb{C}^n$ as $\mathcal{V}(f) \subset \mathbb{C}^N$. Given a witness set for some algebraic set $Z \subset \mathbb{C}^N$ and a system of polynomials $f : \mathbb{C}^N \to \mathbb{C}^n$, the algorithms of this paper compute a numerical irreducible decomposition of the set $Z \cap \mathcal{V}(f)$. While extending the types of intersection problems that can be solved via numerical algebraic geometry, this approach is also a unification of two existing algorithms: the diagonal intersection algorithm and the homotopy membership test. The new approach includes as a special case the "extension problem" where one wishes to intersect an irreducible component A of $\mathcal{V}(g(x))$ with $\mathcal{V}(f(x, y))$, where f introduces new variables, y. For example, this problem arises in computing the singularities of A when the singularity conditions are expressed in terms of new variables associated to the tangent space of A. Several examples are included to demonstrate the effectiveness of our approach applied in a variety of scenarios.

1. Introduction

Numerical algebraic geometry concerns the solution of systems of polynomial equations using numerical methods, principally homotopy methods, also known as polynomial continuation. We take the ground field to be the complex numbers $\mathbb{C}$: the continuity and algebraic completeness of $\mathbb{C}$ are essential for our algorithms. Given a polynomial system $F : \mathbb{C}^N \to \mathbb{C}^n$:

$$F(z_1, \ldots, z_N) = \begin{bmatrix} F_1(z_1, \ldots, z_N) \\ \vdots \\ F_n(z_1, \ldots, z_N) \end{bmatrix}, \quad (1)$$

one wishes to describe its solution set, called the algebraic set defined by F, denoted as

$$\mathcal{V}(F) := \{z \in \mathbb{C}^N \mid F(z) = 0\}.$$

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Early in the development of polynomial continuation, all attention was focused on finding just the isolated solutions of a given polynomial system. In these methods, a homotopy is constructed along with a set of starting solutions. A homotopy path emanates from each start point, and the set of all endpoints of these paths is guaranteed to include all the isolated solutions of the target polynomial system. See, for example, [30] for a treatise from the early era. Major advances in methods for constructing homotopies with fewer paths, which thereby generally reduce the computational cost of numerically tracking the paths, include multi-homogeneous formulations [31], linear-product homotopies [52], and polyhedral homotopies [22,53]. The polyhedral methods make maximal use of whatever sparsity exists in the list of monomials appearing in the polynomials. While this gives the least number of homotopy paths among all the known single-shot homotopy methods, the homotopy construction phase depends on the computation of the mixed volume of the system [25,28], potentially a computationally complex problem.

Numerical algebraic geometry was founded [46] on the recognition that often one may be interested in positive-dimensional solution sets (curves, surfaces, etc.). A methodology was developed to discover such sets by applying polynomial continuation to find witness points on the sets, these points having been isolated by intersecting the positive dimensional set with a generic linear space of complementary dimension. This idea lead to the concept of a *witness set*, to be described precisely later, now considered the fundamental construct of the field. The computation of a numerical irreducible decomposition, first accomplished in [40], produces a witness set for each irreducible component of $\mathcal{V}(F)$. A concise review of the main concepts and vocabulary of the field, such as witness sets, irreducible components, and the numerical irreducible decomposition, are included here in Section 2; for more complete treatments, see [8,47].

Software available for computing just isolated solutions includes Hom4PS [25], HomPack [33], and POLSYS_GLP [49]. Packages that compute both isolated solutions and witness sets for positive-dimensional solution sets are PHCpack [51] and Bertini [7]. The algorithms reported here have been tested using extensions to Bertini.

Soon after the crystallization of the witness set idea, it was seen that it can be advantageous to solve systems in stages. The diagonal intersection methods of Sommese et al. [38,39] compute the intersection of a pair of algebraic sets, each given by a witness set. Using this approach, one may introduce one-by-one the hypersurfaces defined by the individual polynomials in the system, maintaining a numerical irreducible decomposition of the intersection at each stage [44]. This kind of idea was given a new twist with the invention of regeneration methods [17,18]. Both of these methods can be streamlined if one only wants the isolated solutions of the final system. As detailed in [20], to obtain a full irreducible decomposition, including positive-dimensional sets, the diagonal methods have to be completed using isosingular theory [19], based on Thom–Boardman singularities [3,10].

The new algorithm described in this paper generalizes the diagonal intersection problem and makes use of regeneration to solve it. The main problem we solve is to intersect a pure-dimensional algebraic set, $Z \subset \mathbb{C}^N$, given by a witness set, with $\mathcal{V}(F)$, for a polynomial system F as in (1). The result is a numerical irreducible decomposition of $Z \cap \mathcal{V}(F)$. Variants include the case when $Z = A \times B$ and we begin with witness sets for algebraic sets $A \subset \mathbb{C}^M$ and $B \subset \mathbb{C}^N$. In particular, given a witness set for $A \subset \mathcal{V}(G(x))$, this allows us to extend the solution to $(A \times \mathbb{C}^N) \cap \mathcal{V}(F(x, y))$, where $F : \mathbb{C}^M \times \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system that involves new variables, y, not present in $G(x)$. Since our approach extends from A using regeneration, we call it *regeneration extension*. Our examples aim to show the utility of these new capabilities. At the same time, we note the unifying nature of this formulation, as both the existing diagonal homotopy method and the existing homotopy membership test can be viewed as special cases.

The enhanced ability to solve a variety of intersection problems gives great flexibility in how one may approach a problem and often has a large impact on the computational cost of doing so. It is quite common for systems arising in applications and in mathematical research to have solution sets composed of several irreducible components. Having computed a numerical irreducible decomposition for a system under study, the analyst often finds that some of the components are of further interest while it is desired to exclude the others from further investigation. This is because often some components are degenerate or nonphysical in some sense specific to the problem at hand. Our method allows one to compute intersections with just the interesting components while ignoring the rest. Our examples will illustrate how this can sometimes led to considerable savings.

Once a witness set is available for an algebraic set, one may wish to investigate it further by finding points on it that satisfy extra conditions. One possibility is to find its singularities by intersecting the set with the conditions for a point on the set to have a tangent space of higher dimension than the set itself. With our new algorithm, one is free to write the singularity conditions in terms of new variables associated to the tangent directions rather than relying exclusively on determinants of matrices of partial derivatives in the original variables. The additional polynomials might instead define other "critical" conditions on the set, such as computing critical points of the distance to a fixed point, e.g., [4,14,36] (see also [12]), and critical points of a projection, e.g., [9,29]. We will also illustrate our algorithm at work on this kind of problem.

Another application of the new algorithm arises in finding exceptional sets by taking sequences of fiber products, as in [48]. Not only does each new fiber product introduce a new set of variables, but also symmetry groups act on the irreducible components that arise. The ability to pick out one component for further investigation avoids the combinatorial explosion of fiber products that would otherwise ensue. This topic is too complicated to take up here, but it will be addressed in a forthcoming paper. For now, we simply note that the kind of flexibility provided by the algorithm given here is crucial to tackling any but the simplest problems concerning exceptional sets.