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A multigrid solution for the Cahn–Hilliard equation on nonuniform grids

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ABSTRACT

We present a nonlinear multigrid method to solve the Cahn–Hilliard (CH) equation on nonuniform grids. The CH equation was originally proposed as a mathematical model to describe phase separation phenomena after the quenching of binary alloys. The model has the characteristics of thin diffusive interfaces. To resolve the sharp interfacial transition, we need a very fine grid, which is computationally expensive. To reduce the cost, we can use a fine grid around the interfacial transition region and a relatively coarser grid in the bulk region. The CH equation is discretized by a conservative finite difference scheme in space and an unconditionally gradient stable type scheme in time. We use a conservative restriction in the nonlinear multigrid method to conserve the total mass in the coarser grid levels. Various numerical results on one-, two-, and three-dimensional spaces are presented to demonstrate the accuracy and effectiveness of the nonuniform grids for the CH equation.

1. Introduction

We consider a nonlinear multigrid solution for the following Cahn–Hilliard (CH) equation on nonuniform grids:

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \Delta\mu(\phi(\mathbf{x},t)), \quad \mathbf{x}\in\Omega,\ t>0, \tag{1}$$

$$\mu(\phi(\mathbf{x},t)) = F'(\phi(\mathbf{x},t)) - \epsilon^2\Delta\phi(\mathbf{x},t), \tag{2}$$

where $\Omega \subset \mathrm{R}^d$ $(d = 1, 2, 3)$, ϕ is a conserved scalar field, $F(\phi) = 0.25(\phi^2 - 1)^2$, and ϵ is a positive constant. The CH equation was introduced to describe phase separation phenomena [1]. This equation arises from the Ginzburg–Landau free energy

$$\mathcal{E}(\phi) = \int_\Omega \left(F(\phi) + \frac{\epsilon^2}{2}|\nabla\phi|^2\right)d\mathbf{x}. \tag{3}$$

The natural and no-flux boundary conditions are

$$\mathbf{n}\cdot\nabla\phi = \mathbf{n}\cdot\nabla\mu = 0 \text{ on } \partial\Omega,\ \text{ where } \mathbf{n} \text{ is a normal vector to } \partial\Omega. \tag{4}$$

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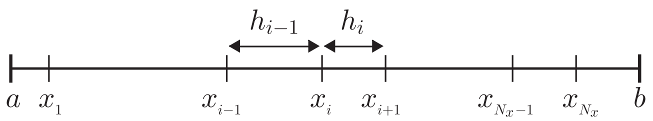

Fig. 1. Discretization of the domain with a nonuniform grid.

Then, we have

$$\frac{d}{dt}\mathcal{E}(\phi) = -\int_{\Omega} |\nabla\mu|^2 d\mathbf{x}, \tag{5}$$

$$\frac{d}{dt}\int_{\Omega} \phi d\mathbf{x} = 0, \tag{6}$$

where we used the no flux boundary condition Eq. (4). Therefore, the total energy is non-increasing and the total mass is conserved in time.

Most existing finite difference methods for the CH equation employed the uniform grid [2–8] or adaptive mesh refinement [9–11]. There are some cases which need non-square domains or adaptive mesh grids. Generally, adaptive mesh refinement technique is very complex to implement and even it is extremely difficulty to incorporate fluid flows into the CH equation. Various studies were performed on nonuniform grids to solve the Poisson [12,13], Euler [14,15], and Navier–Stokes [16,17] equations. However, to the authors' knowledge, the CH equation has not been solved using a multigrid method on nonuniform grids. Therefore, the main purpose of the present paper is to present a multigrid solution for solving the CH equation on nonuniform grids.

The remainder of this paper is organized as follows. In Section 2, we describe the numerical solution algorithm. The numerical results demonstrating the performance of the proposed algorithm on nonuniform grids are presented in Section 3. Finally, Section 4 gives our conclusions.

2. Numerical solution

2.1. Discretization

In this section, we discretize the CH equation on a nonuniform grid. For simplicity of exposition, we discretize the CH Eqs. (1) and (2) in one-dimensional space, i.e., $\Omega = (a, b)$. Two- and three-dimensional discretizations are defined analogously. Let x_i be the nonuniform grid points, that is, $x_i = x_{i-1} + h_{i-1}$ for $1 < i \le N_x$, where N_x is a positive even integer and h_i is the nonuniform grid-spacing as shown in Fig. 1.

Let ϕ_i^n and μ_i^n be approximations of $\phi(x_i, t_n)$ and $\mu(x_i, t_n)$, respectively. Here $t_n = (n-1)\Delta t$ and Δt is the time step. We discretize Eqs. (1) and (2) using the semi-implicit scheme and the nonlinear splitting algorithm [18]:

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \Delta_d \mu_i^{n+1}, \quad 1 \le i \le N_x, \quad n \ge 0, \tag{7}$$

$$\mu_i^{n+1} = (\phi_i^{n+1})^3 - \phi_i^n - \epsilon^2 \Delta_d \phi_i^{n+1}, \tag{8}$$

where the discrete Laplacian is defined as $\Delta_d\phi_i = 2(\phi_{i+1} - \phi_i)/[h_i(h_i + h_{i-1})] - 2(\phi_i - \phi_{i-1})/[h_{i-1}(h_i + h_{i-1})]$. Here $h_0 = 2(x_1 - a)$ and $h_{N_x} = 2(b - x_{N_x})$. The Neumann boundary condition Eq. (4) is implemented as $\phi_0 = \phi_1$ and $\phi_{N_x+1} = \phi_{N_x}$. Let $\boldsymbol{\phi}^n = (\phi_1^n, \phi_2^n, \ldots, \phi_{N_x}^n)$ and $\mu^n = (\mu_1^n, \mu_2^n, \ldots, \mu_{N_x}^n)$. We define the discrete l_2-norm as

$$\|\boldsymbol{\phi}\|_2 = \sqrt{\sum_{1\le i\le N_x} \phi_i^2 \frac{h_{i-1} + h_i}{2}}. \tag{9}$$

We define the discrete energy functional by

$$\mathcal{E}_d(\boldsymbol{\phi}^n) = \sum_{i=1}^{N_x} F(\phi_i^n)\frac{h_{i-1} + h_i}{2} + \frac{\epsilon^2}{2}\sum_{i=1}^{N_x-1} \frac{(\phi_{i+1}^n - \phi_i^n)^2}{h_i}. \tag{10}$$

We also define the discrete total mass as

$$\mathcal{M}_d(\boldsymbol{\phi}^n) = \sum_{i=1}^{N_x} \phi_i^n \frac{h_{i-1} + h_i}{2}. \tag{11}$$